Akashi et al., Int J Comput Softw Eng 2023, 8: 188
https://doi.org/10.15344/2456-4451/2023/188

International Journal of
Computer & Software Engineering

Original Article Open Access

# On the Possibility of Multidimensional Data Compression Based on the Hilbert's 13th Problem

**Shigeo Akashi*, Toyonari Iwata and Yao Tong**
*Department of Information Sciences, Tokyo University of Science, Noda City, Chiba Prefecture, 278-8510, Japan*

## Abstract

In this paper, we discuss the problem asking if Kolmogorov-Arnold representation theorem, which has played so important roles in solving Hilbert's 13th problem, can be applied to the theory of multidimensional numerical data compression, because the way of several time nested superposition being used in Kolmogorov-Arnold representation theorem is analogous to the way of construting a multidimensional numerical table with several fewer dimensional numerical tables. Exactly speaking, we discuss two versions of the original Hilbert's 13th problem, which can be derived from the replacement of the condition of continuous functions with the condition of the functions being differentiabie in all directions.

**Publication History:**

Received: December 04, 2023
Accepted: December 26, 2023
Published: December 28, 2023

**Keywords:**

Data compression, Kolmogorov-Arnold representation, Continuous function

## Introduction

From the mathematical point of view, the dimension-reducing numerical data compression is to the theory of data compression what several time nested superposition is to Hilbert's 13th problem [1, 2, 6, 8], If the solution to Hilbert's 13th problem is applied to continuous functions of two real variables, then it can be proved that there exists two families of five monotone-increasing continuous functions of one real variable $\{\phi_i(\cdot); 0 \le i \le 4\}$ and $\{\psi_i(\cdot); 0 \le i \le 4\}$ satisfying that, for any continuous function $f(\cdot, \cdot)$ of two real variables, there exists an appropriate continuous function $g_f(\cdot)$ of one real variable, depending only on $f(\cdot, \cdot)$ and enabling $f(\cdot, \cdot)$ to be represented as the following:

$$f(x_1,x_2)=\sum_{i=0}^{4}g_f(\phi_i(x_1)+\psi_i(x_2)),\ x_1,x_2\in$$

This resultant formula, which is called Kolmogorov-Arnold representation [4, 5, 7], can derive another problem from itself, asking whether or not any continuous function $f(\cdot, \cdot)$ of two real variables can be represented as more simplified formulae than the above formula such as the following:

$$f(x_1,x_2)=\sum_{i=0}^{3}g_f(\phi_i(x_1)+\psi_i(x_2)),\ x_1,x_2\in\mathbb{R}$$

$$f(x_1,x_2)=\sum_{i=0}^{2}g_f(\phi_i(x_1)+\psi_i(x_2)),\ x_1,x_2\in\mathbb{R}$$

For example, it is clear that we cannot find any monotone-increasing continuous functions $\phi_0(\cdot)$ and $\psi_0(\cdot)$ satisfying that, for any continuous function $f(\cdot, \cdot)$, there exists an appropriate continuous function $g_f(\cdot)$ of one variable, depending only on $f(\cdot, \cdot)$ and enabling $f(\cdot, \cdot)$ to be represented as the following:

$$f(x_1,x_2)=g_f(\phi_0(x_1)+\psi_0(x_2)),\ x_1,x_2\in\mathbb{R}$$

because, for any sufficiently small positive number $\Delta x_1$, there exists an appropriate positive number $\Delta x_2$ satisfying the following:

$$\phi_0(x_1)+\psi_0(x_2)=\phi_0(x_1+\Delta x_1)+\psi_0(x_2-\Delta x_2)$$

This equality implies that, for any function $f(\cdot, \cdot)$, the following equalities:

$$\begin{aligned} f(x_1,x_2) &= g_f(\phi_0(x_1)+\psi_0(x_2)) \\ &= g_f(x_1+\Delta x_1)+\psi_0(x_2-\Delta x_2) \\ &= f(x_1+\Delta x_1, x_2-\Delta x_2) \end{aligned}$$

hold, and eventually, the above equalities lead us to such a contradiction that, for any funition $f(\cdot, \cdot)$ of two variables, the value of $(x_1, x_2)$ under the function $f(\cdot, \cdot)$ should be equal to the value of $(x_1+\Delta x_1, x_2-\Delta x_2)$ under tha same function.

Exactly speaking, When the differentiability condition is discussed in the teory of the functions of multivariables, this condition can be classifined into two more accurate differentiability ones, one of which is the differentiability in all directions and the other of which is the total differentiability. It is clear that, if any function of two variables is totally differentiable, then the function is necesarily differentiable in all directions. Actually, the converse implication does not hold. While the total differentiability plays so important roles in the theory of the functions of multivariables, the differentiability in all directions does the same important roles as the total differentiability in the theory of multidimensional numerical data compression.

In the former half of this paper and in the latter half, we discuss the problem asking whether Kolmogorov-Arnold representation theorem holds or not in case of the set of all one-time differentiable functions in all directions and in case of the set of all two-time differentiable functions in all directions, respectively, because the total differentiability condition, which has been usually assumed in the theory of the functions of multi-variables, is too strong to apply to the theory of multi-dimensional numerical data compuression.

*Corresponding Author: Prof. Shigeo Akashi, Department of Information Sciences, Tokyo University of Science, Noda City, Chiba Prefecture, 278-8510, Japan, E-mail: hanada@cc.saga-u.ac.jp

**Citation:** Akashi S, Iwata T, Tong Y (2023) On the Possibility of Multidimensional Data Compression Based on the Hilbert's 13th Problem. Int J Comput Softw Eng 8: 188. doi: https://doi.org/10.15344/2456-4451/2023/188

**Citation:** Akashi S, Iwata T, Tong Y (2023) On the Possibility of Multidimensional Data Compression Based on the Hilbert's 13th Problem. Int J Comput Softw Eng 8: 188. doi: https://doi.org/10.15344/2456-4451/2023/188



## The case of functions of two variables being one-time differentiable in all directions

In this section, we discuss the problem asking if a version of Kolmogorov-Arnold representation in case of functions of two variables which can be one time differentiable in all directions holds. This version can be formulated as the following:

**Propositon 1**. For any positive integer n, there does not exist n pairs being composed of two strictly increasing one-time differentiable functions of one variable $\{\{\phi_i(\cdot),\ \psi_i(\cdot)\};\ 0 \le i \le n-1\}$ satisfying that,or any function of two variables $f(\cdot,\cdot)$ which can be one-time differentiable in all directions, there exists a one-time differentiable function gf $(\cdot)$ of one variable enabling $f(\cdot,\cdot)$ to be represented as the following:

$$f(x_1,x_2)=\sum_{i=0}^{n=1} g_f(\phi_i(x_1)+\psi_i(x_2)),\ x_1,x_2 \in \mathbb{R}$$

**Proof.** Under the assumptions stated in this proposition, we can obtain the following equalities:

$$\frac{\partial f}{\partial x_1}=\sum_{i=0}^{n-1} g_f'(\phi_i(x_1)+\psi_i(x_2))\ \phi_i' x_1,$$

$$\frac{\partial f}{\partial x_2}=\sum_{i=0}^{n-1} g_f'(\phi_i(x_1)+\psi_i(x_2))\ \psi_i' x_2.$$

Then, Taylor expansion assures that, for any sufficiently small numbers $\Delta x_1$ and $\Delta x_2$, the following equalities hold:

$$\begin{aligned}
f(x_1+\Delta x_1, x_2+\Delta x_2) &= \sum_{i=0}^{n-1} g_f(\phi_i(x_1+\Delta x_1)+\psi_i(x_2+\Delta x_2)) \\
&= \sum_{i=0}^{n-1} g_f(\{\phi_i(x_1)+\psi_i(x_2)\}+\{\phi_i'(x_1)\Delta x_1+\psi_i'(x_2)\Delta x_2\}+0(|\Delta x_1|+|\Delta x_2|)) \\
&= \sum_{i=1}^{n-1} g_f(\phi_i(x_1)+\psi_i(x_2)) + \sum_{i=0}^{n-1} g_i'(\phi_i(x_1)+\psi_i(x_2))+(\{\phi_i'(x_1)\Delta x_1+\psi_i'(x_2)\Delta x_2\}+0(|\Delta x_1|+|\Delta x_2|)) \\
&= f(x_1,x_2)+\frac{\partial f}{\partial x_1}\Delta x_1+\frac{\partial f}{\partial x_2}\Delta x_2+0(|\Delta x_1|+|\Delta x_2|))
\end{aligned}$$

Actually, these equalities derive a contradiction that all the one time differentiable functions of two variables are totally differentiable. Therefore, we can conclude the proof.

**Remark 2**. Here is an example of the function which is not totally differentiable but differentiable in all directions. Let $f(\cdot,\cdot)$ be the function which is defined as the following:

$$\begin{cases} f(x_1,x_2)=\dfrac{x_1x_2^2}{x_1^2+x_2^2},(x_1,x_2)\neq(0,0), \\ f(0,0)=0. \end{cases}$$

Then, the following equalities hold:

$$\left.\frac{\partial f}{\partial x_1}\right|_{x_1=0,x_2=0}=\left.\frac{\partial f}{\partial x_2}\right|_{x_1=0,x_2=0}=0.$$

We can display the graphical image of this function as the following:

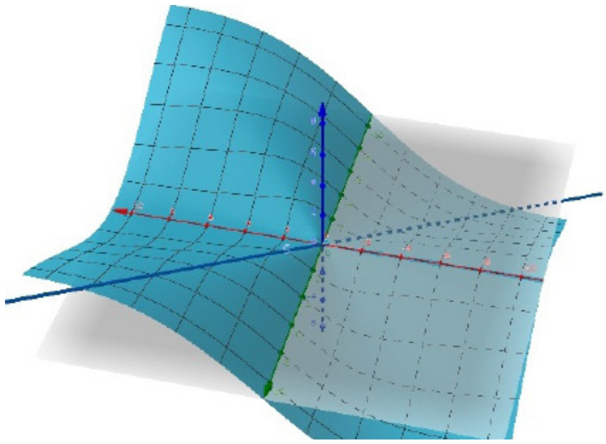

Figure 1. Graphic image of $z=\dfrac{x_1x_2^2}{x_1^2x_2^2}$.

Let $\theta_0$ be a positive number which is less than $\pi/2$ satisfying $\cos\theta_0 = 3/5$. Then, for any positive number $r$, we can obtain the following equality:

$$f(r\cos\theta_0, r\sin\theta_0)=\frac{48r}{125}.$$

Therefore, this equality derives to the following equality:

$$\lim_{r\to 0}\frac{f(r\cos\theta_0, r\sin\theta_0)-f(0,0)}{r}=\frac{48}{125}$$

This equality shows that the given function is not totally differentiable.

## The case of functions of two variables being two-time differentiable in all directions

Throughout this section, we use the same definitions and notations as used in the previous section, and we discuss the problem asking if a version of Kolmogorov-Arnold representation in case of two-time totally differentiable functions of two variables holds. This version can be formulated as the following:

**Propositon 3**. For any positive integer n, there does not exist n pairs being composed of two strictly increasing two-time differentiable functions of one variable $\{\{\phi_i(\cdot),\ \psi_i(\cdot)\};\ 0 \le i \le n-1\}$ satisfying that, for any totally differentiable function of two variables $f(\cdot,\cdot)$, there exists a two-time differentiable function of one variable enabling $f(\cdot,\cdot)$ to be represented as the following:

**Citation:** Akashi S, Iwata T, Tong Y (2023) On the Possibility of Multidimensional Data Compression Based on the Hilbert's 13th Problem. Int J Comput Softw Eng 8: 188. doi: https://doi.org/10.15344/2456-4451/2023/188



$$f(x_1, x_2) = \sum_{i=0}^{n-1} g_f(\phi_i(x_1) + \psi_i(x_2)), \ \ x_1, x_2 \in \mathbb{R}.$$

Proof. Let $f(\cdot, \cdot)$ be the function defined as the following:

$$\begin{cases} f(x_1, x_2) = \dfrac{x_1 x_2^3}{x_1^2 + x_2^2}, (x_1, x_2) \neq (0, 0), \\ f(0, 0) = 0. \end{cases}$$

Then, we can obtain the following partial differentials:

$$\frac{\partial f}{\partial x_1} = \frac{x_2^5 - x_1^2 x_2^3}{(x_1^2 + x_2^2)^2}$$

$$\frac{\partial f}{\partial x_2} = \frac{3x_2^3 x_2^2 - x_1 x_2^4}{(x_1^2 + x_2^2)^2}$$

and

$$\left.\frac{\partial f}{\partial x_1}\right|_{x_1=0} = \lim_{\Delta x_1 \to 0} \frac{\dfrac{\Delta x_1 x_2^3}{\Delta x_1^2 + x_2^2} - 0}{\Delta x_1} = x_2$$

$$\left.\frac{\partial f}{\partial x_2}\right|_{x_2=0} = \lim_{\Delta x_2 \to 0} \frac{\dfrac{x_1 \Delta x_2^3}{x_1^2 + \Delta x_2^2} - 0}{\Delta x_2} = 0$$

These equalities imply the following equalities:

$$\lim_{\Delta x_2 \to 0} \frac{\left.\dfrac{\partial f}{\partial x_1}\right|_{x_1=0, x_2=\Delta x_2} - \left.\dfrac{\partial f}{\partial x_1}\right|_{x_1=0, x_2=0}}{\Delta x_2} = \lim_{\Delta x_2 \to 0} \frac{\Delta x_2 - 0}{\Delta x_2} = 1$$

$$\lim_{\Delta x_1 \to 0} \frac{\left.\dfrac{\partial f}{\partial x_2}\right|_{x_1=0, x_2=0} \quad \left.\dfrac{\partial f}{\partial x_2}\right|_{x_1=0, x_2=0}}{\Delta x_1} = \lim_{\Delta x_2 \to 0} \frac{0 \quad 0}{\Delta x_1} = 0$$

On the contrary, it follows from the assumptions used in Proposition 3 that we can obtain the following equalities:

$$\frac{\partial f}{\partial x_2 \partial x_1} = \sum_{i=0}^{n-1} g_f''(\phi_i(x_1) + \psi_i(x_2))\phi_i'(x_1)\psi_i'(x_2)$$
$$= \frac{\partial^2 f}{\partial x_1 \partial x_2}$$

Actually, the above equalities conclude the following contradiction:

$$1 = \left.\frac{\partial^2 f}{\partial x_2 \partial x_1}\right|_{x_1=0, x_2=0} = \left.\frac{\partial^2 f}{\partial x_1 \partial x_2}\right|_{x_1=0, x_2=0} = 0$$

**Remark 4.** We can display the graphical image of the function used in Proposition 3 as the following:

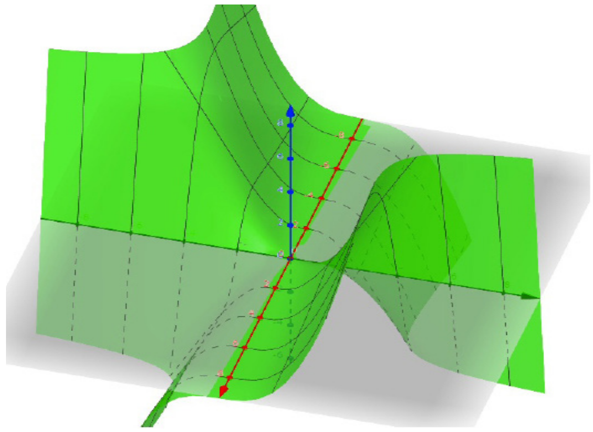

Figure 2. Graphic image of $z = \dfrac{x_1 x_2^3}{x_1^2 x_2^2}$

## Conclusions

If there exists an algorithm which enables two dimensional numerical data to be compressed into several one dimensional numerical sequences, then the algorithm would contribute to saving the memory area of all the computers, whether its way of reproduction is invertible or non-invertible. Actually, Hilbert's 13th problem may tell us that it seems to be difficult to develop such a skill as dimension-reducing numerical data compression.

## Competing Interests

The author declare that he has no competing interests.

## References

1. Akashi S (1990) The asymptotic behavior of ε-entropy of a compact positive operator. Journal ofMathematical Analysis and Applications 153: 150-157.
2. Akashi S (2003) A version of Hilbert's 13th problem for analytic functions. The Bulletin of the London Mathematical Society 35: 8-14.
3. Babenko KI (1985) On the best approximation of a class of analytic functions, Izv. 22: 631-640.
4. Erohin VD (1958) On the asymptotic behavior of the ϵ-entropy of analytic functions, Dokl., 120: 949-952.
5. Kolmogorov AN (1957) On the representation of continuous functions of several variables by superpositions of continuous functions of one variable and addition, Dokl., 114: 679-681.
6. Lorentz GG (1966) Approximation of Functions, Holt, Rinehart and Winston Inc., New York.
7. Rudin W (1987) Real and Complex Analysis, McGraw-Hill Book Company, Third Edition, New York,
8. Vitushkin AG (1964) Some properties of linear superpositions of smooth functions, Dokl., 156: 1003-1006.