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On Fractional Taylor's Formula and Fractional Cauchy' Formula with Multivariate

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Abstract

In this paper, a generalized fractional Taylor's formula and Cauchy's formula of the kind

$$f(x,y)=\sum_{j=0}^{n}\frac{D^{j\alpha}f(x_0,y_0)}{\Gamma(j\alpha+1)}+R_n^{\alpha}(x,y),\quad \frac{f(x,y)-\sum_{j=0}^{n}\frac{D^{j\alpha}f(x_0,y_0)}{\Gamma(j\alpha+1)}}{g(x,y)-\sum_{j=0}^{n}\frac{D^{j\alpha}f(x_0,y_0)}{\Gamma(j\alpha+1)}}=\frac{R_n^{\alpha}(x,y)}{T_n^{\alpha}(x,y)}$$

where $0<\alpha\le 1$, is established. Such expression is precisely the classical Taylor's formula and Cauchy's formula in the particular case $\alpha = 1$. In addition, detailed expressions for $R_n^{\alpha}(x,y)$ and $T_n^{\alpha}(x,y)$ involving the sequential Caputo fractional derivative are also given.

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Introduction

The ordinary Taylor's formula has been generalized by many authors. Riemann [1] had already written a formal version of the generalized Taylor's series:

$$f(x+h)=\sum_{m=-\infty}^{\infty}\frac{h^{m+r}}{\Gamma(m+r+1)}\left(J_a^{m+r}f\right)(x)\quad (1.1)$$

where J^{m+r} is the Riemann-Liouville fractional integral of order m+r. Afterwards, Watanabe [2] obtained the following relation:

$$f(x)=\sum_{k=-m}^{n-1}\frac{(x-x_0)^{\alpha+k}}{\Gamma(\alpha+k+1)}\left(D_a^{m+r}f\right)(x_0)+R_{n,m}\quad (1.2)$$

with $m<\alpha$, $a\le x_0<x$, and

$$R_{n,m}=(I_{x_0}^{\alpha+n}D_{\alpha}^{\alpha+n}f)(x)+\frac{1}{\Gamma(\alpha-m)}\int_{\alpha}^{x_0}(x-t)^{-\alpha-m-1}(D_{\alpha}^{\alpha-m-1}f)(t)dt$$

where $D^{\alpha+n}$ is the Riemann-Liouville fractional derivative of order $\alpha+n$.

On the other hand, a variant of the generalized Taylor's series was given by Dzherbashyan and Nersesyan [3]. For *f* having all of the required continuous derivatives, they obtained

$$f(x)=\sum_{k=o}^{m-1}\frac{\left(D^{(\alpha k)}f\right)(0)}{\Gamma(1+\alpha_k)}x^{\alpha k}+\frac{1}{\Gamma(1+\alpha_m)}\int_0^x(x-t)^{\alpha_m-1}\left(D^{(\alpha_m)}f\right)(t)dt\quad (1.3)$$

where $0<x,\alpha_0,\alpha_1,\dots,\alpha_m$ is an increasing sequence of real numbers such that $0<\alpha_k-\alpha_{k-1}\le 1, k=1,\dots,m$ and $D^{(\alpha_m)}f=I_0^{1-(\alpha_k-\alpha_{k-1})}D_0^{1+\alpha_{k-1}}f$

Under certain conditions for f and $\alpha \in [0, 1]$, Trujillo et al. [4] introduce the following generalized Taylor's formula:

$$f(x)=\sum_{j=o}^{n}\frac{c_j(x-a)^{(j+1)\alpha-1}}{\Gamma((j+1)\alpha)}+R_n(x,a)\quad (1.4)$$

$$R_n(x,a)=\frac{(D_a^{(n+1)\alpha}f)(\xi)}{\Gamma((n+1)\alpha+1)}\cdot(x-1)^{(n+1)\alpha},a\le\xi\le x,$$

$$c_j=\Gamma(a)[(x-a)^{1-\alpha}D_a^{j\alpha}f](a+),j=0,1,\dots,n$$

and the sequential fractional Riemann-Liouville derivative is denoted by $D_a^{n\alpha}=D_a^{\alpha}\cdot D_a^{\alpha}\cdot\dots\cdot D_a^{\alpha}(n-times)$

Recently, Zaid M. Odibat, Nabil T. Shawagfeh [5] obtain a new generalized Taylor's formula of this kind

$$f(x)=\sum_{j=o}^{n}\frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)}(D_{\alpha}^{j\alpha}f)(a)+\frac{(D_{\alpha}^{j\alpha}f)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha}\quad (1.5)$$

with $a\le\xi\le x$; where $D^{j\alpha}$ is the sequential fractional Caputo derivative.

To the best of our knowledge, the recent developments on Fractional Taylor's Formula and Fractional Cauchy' Mean Value Theorem with Multivariate is not well reported. G. Jumarie [6] had given the following Multivariate fractional Taylor Series

$$f(x+h,y+l)=E_{\alpha}\left(h^{\alpha}D_x^{\alpha}\right)E_{\alpha}\left(l^{\alpha}D_y^{\alpha}\right)f(x,y)=$$

$$E_{\alpha}\left(l^{\alpha}D_y^{\alpha}\right)E_{\alpha}\left(h^{\alpha}D_x^{\alpha}\right)f(x,y)=E_{\alpha}\left[\left(hD_x+lD_y\right)^{\alpha}\right]f(x,y)\quad (1.6)$$

where $E_\alpha(x)$ denotes the Mittage-Leer function dened by the expression [7-15]

$$E_{\alpha}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)}$$

But I am afraid Eq.(1.6) is incorrect, since its proof is based on the following equality

$$E_{\alpha}[(u+\upsilon)^{\alpha}]=E_{\alpha}(u^{\alpha})E_{\alpha}(\upsilon^{\alpha})\quad (1.7)$$

which seems incorrect unless $\alpha = 1$.

In order to establish fractional Taylor's formula and fractional Cauchy' Mean value theorem, the main diculty seems that how to give the suitable denition of fractional integral and derivative of function with multivariate.

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In this paper, we will give a new denition of fractional integral and derivative of function with multivariate, derive fractional Taylor's formula with multivariate (a) with the Lagrange remainder term; (b) with integral remainder term. and derive fractional Cauchy formula with multivariate (a) with the Lagrange remainder term; (b)with integral remainder term.

As far as we are aware, this denition and results have not been published elsewhere previously.

Denitions and Properties

For the concept of fractional derivative we will adopt Caputo's denition which is a modication of the Riemann-Liouville denition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the eld variables and their integer order which in the case in most physical processes. More detailed information on fractional calculus may be found in these books [7-14].

Definition

A function f(x)(x > 0) is said to be in the space $C_\alpha(\alpha \in R)$ if it can be written as $f(x) = x^p f_1(x)$ for p > where $f_1(x)$ is continuous in $(0,\infty)$ and it is said to be in the space if $f^{(m)} \in C\alpha$, m ∈ N.

Definition

[7-14] Let f(x) ∈ $C_\alpha(a, \infty)$, the Riemann-Liouville integral operator of order α > 0 is dened as

$$(D_a^{-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a$$

Definition

[7-14] Let f(x) ∈ $C_\alpha^{(m)}$ the Caputo fractional derivative of f(x) of order α> 0 is dened as

$$(D_a^\alpha f)(x) = (D_a^{\alpha-m} f^{(m)})(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}} dt$$

for m -1 < α ≤ m,m ∈ N, x≥a

Definition

[7-14] Let $D^{-\alpha}$ be Riemann-Liouville integral operator of order α > 0, be Caputo fractional derivative operator of order α > 0, 0 <α ≤ 1, $D_a^\alpha f(x) \in C(a,b)$ then

$$[D_a^{-\alpha} D_a^\alpha f](x) = f(x) - f(a)$$

In order to derive fractional Taylor's formula and Cauchy formula of a function with multivariate, we will rst give the following integrate denition of a function f(x; y), (x; y) ∈ D, where D is a convex domain.

Definition

Let $(x_0, y_0), (x, y) \in D, \Delta x = x - x_0, \Delta y = y - y_0, 0 \le s \le 1$ define

$$(D^{-1} f)(x_0 + s\Delta x, y_0 + s\Delta y) = \int_0^s f(x_0 + t\Delta x, y_0 + t\Delta y)dt \qquad (2.1)$$

when s = 1, define

$$(D^{-1} f)(x, y) = \int_0^1 f(x_0 + t\Delta x, y_0 + t\Delta y)dt \qquad (2.2)$$

Proposition

Let k ∈ N $(x_0, y_0), (x, y) \in D, \Delta x = x - x_0, \Delta y = y - y_0, 0 \le s \le 1$ then

$$(D^{-k} f)(x_0 + s\Delta x, y_0 + s\Delta y) = \frac{1}{(k-1)!} \int_0^s (s-t)^{k-1} f(x_0 + t\Delta x, y_0 + t\Delta y)dt \qquad (2.3)$$

Proof By Denition we have

$$(D^{-2} f)(x_o + s\Delta x, y_0 + s\Delta y) = \int_0^s D^{-1} f(x_0 + t\Delta x, y_0 + t\Delta y)dt$$

$$= \int_0^s dt \int_0^t f(x_0 + u\Delta x, y_0 + u\Delta y)du$$

$$= \int_0^s du \int_u^s f(x_0 + u\Delta x, y_0 + u\Delta y)dt = \int_0^s (s-u) f(x_0 + u\Delta x, y_0 + u\Delta y)du$$

By induction, it is not hard to prove that

$$(D^{-k} f)(x_o + s\Delta x, y_0 + s\Delta y) = \frac{1}{(k-1)!} \int_0^s (s-t) f(x_0 + t\Delta x, y_0 + t\Delta y)dt$$

Now we can dene fractional integral of f(x; y) of order γ

Definition

Let $\gamma \in R^+, (x_0, y_0), (x, y) \in D, \Delta x = x - x_0, \Delta y = y - y_0, 0 \le s \le 1$ define

$$(D^{-\gamma} f)(x_0 + s\Delta x, y_0 + s\Delta y)$$

$$= \frac{1}{\Gamma(\gamma)} \int_0^s (s-t)^{\gamma-1} f(x_0 + t\Delta x, y_0 + t\Delta y)dt \qquad (2.4)$$

when s = 1, define

$$(D^{-\gamma} f)(x, y) = \frac{1}{\Gamma(\gamma)} \int_0^1 (1-t)^{\gamma-1} f(x_0 + t\Delta x, y_0 + t\Delta y)dt \qquad (2.5)$$

For convenience, Let us set

$\varphi(t) = f(x_0 + t\Delta x, y_0 + t\Delta y)$ then we have

Proposition

Let $(x_0, y_0), (x, y) \in D, \Delta x = x - x_0, \Delta y = y - y_0, 0 \le s \le 1$ then

$$(D^{-\gamma} f)(x_0 + s\Delta x, y_0 + s\Delta y)$$

$$= \frac{1}{\Gamma(\gamma)} \int_0^s (s-t)^{\gamma-1} f(x_0 + t\Delta x, y_0 + t\Delta y)dt = (D^{-\gamma}\varphi)(s) \qquad (2.6)$$

and

$$(D^{-\gamma} f)(x, y) = \frac{1}{\Gamma(\gamma)} \int_0^1 (1-t)^{\gamma-1} f(x_0 + t\Delta x, y_0 + t\Delta y)dt = (D^{-\gamma}\varphi)(1) \qquad (2.7)$$

By Proposition it is easy to see that

Proposition

Let α, β ∈ R^+ then

$$D^{-\alpha} D^{-\beta} f(x_0 + s\Delta x, y_0 + s\Delta y) = D^{-(\alpha+\beta)} f(x_0 + s\Delta x, y_0 + s\Delta y)$$

and

$$D^{-\alpha} D^{-\beta} f(x, y) = D^{-(\alpha+\beta)} f(x, y)$$

Definition

If n ∈ N, define

$$D^n f(x_0 + s\Delta x, y_0 + s\Delta y)$$

$$= (\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial x})^n f(x_0 + s\Delta x, y_0 + s\Delta y) \qquad (2.8)$$

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$$D^n f(x,y) = (\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial x})^n f(x,y) \quad (2.9)$$

Definition

Let μ> 0, and let n be the smallest integer exceeding μ , we define the fractional derivative of f of order μ as following

$$\left(D^{\mu} f\right)(x_0 + s\Delta x, y_0 + s\Delta y) = (D^{\mu-n} D^n f)(x_0 + s\Delta x, y_0 + s\Delta y) \quad (2.10)$$

and

$$\left(D^{\mu} f\right)(x,y) = (D^{\mu-n} D^n f)(x,y) \quad (2.11)$$

From Denition it is easy to know

Proposition

Let $\varphi(s) = f(x_0 + s\Delta x, y_0 + s\Delta y)$ then

$$D^n f(x_0 + s\Delta x, y_0 + s\Delta y) = \varphi^{(n)}(s) \quad (2.12)$$

and $D^n f(x,y) = \varphi^{(n)}(1) \quad (2.13)$

Proposition

Let μ ∈ R^+ then

$$\left(D^{\mu} f\right)(x_0 + s\Delta x, y_0 + s\Delta y) = \left(D^{\mu}\varphi\right)(s) \quad (2.14)$$

and $\left(D^{\mu} f\right)(xy) = \left(D^{\mu}\varphi\right)(1) \quad (2.15)$

Generalized Taylor's mean value theorem and generalized Cauchy's mean value theorem with one Variable

In this section, we will give fractional Taylor's mean value theorem and Cauchy's mean value theorem involving the sequential Caputo fractional derivative with one variable.Let us begin with some basic fractional mean value theorems.

Theorem

(Fractional Lagrange's mean value theorem) Suppose that f(x) ∈ C[a, b] and $D_a^{\alpha} f(x) \in C[a,b]$, for 0≤α≤1, then we have

$$f(b) - f(a) = \frac{1}{\Gamma(\alpha+1)} D_a^{\alpha} f(\xi)(b-a)^{\alpha} \quad (3.1)$$

with a≤ξ≤b

Proof

In view of Proposition we have

$$f(b) - f(a) = \left[D_a^{-\alpha} D_a^{\alpha} f\right](b) = \frac{1}{\Gamma(\alpha)}\int_a^b (b-T)^{\alpha-1}\left[D_a^{\alpha} f\right](T)dT$$

$$= \left[D_a^{\alpha} f\right](\xi)\frac{1}{\Gamma(\alpha)}\int_a^b (b-T)^{\alpha-1} dT = \left[D_a^{\alpha} f\right](\xi)\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}$$

From above Theorem It is easy to obtain

Corollary

(Fractional Rolle's mean value theorem) Suppose that f(x) ∈ C[a,b], $D_a^{\alpha} f(x), D_a^{\alpha} g(x) \in C[a,b]$ for 0≤α≤1, and f(a)=f(b), then there exists ξ ∈ (a,b) such that

$$D_a^{\alpha} f(\xi) = 0 \quad (3.2)$$

Now we can derive fractional Cauchy's mean value theorem with one variable.

Theorem

(Fractional Cauchy's mean value theorem) Suppose that f(x),g(x) ∈ C [a,b] and $D_a^{\alpha} f(x), D_a^{\alpha} g(x) \in C[a,b]$, for 0≤α≤1, then we have

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{D_a^{\alpha} f(\xi)}{D_a^{\alpha} g(\xi)} \quad (3.3)$$

with a≤ξ≤b

Proof

Set F(x) = [f(b)-f(a)][g(x)-g(a)]-[f(x)-f(a)][g(b)-g(a)], then F(a) =F(B) =0 in view of fractional Rolle's mean value Corollary so that there exists ξ ∈ (a, b), such that

$$D_a^{\alpha} F(\xi) = 0$$

Therefore, we have

$$[f(b) - f(a)]D_a^{\alpha} g(\xi) - D_a^{\alpha} f(\xi)[g(b) - g(a)] = 0$$

Theorem is completed.

Theorem

(Fractional Taylor's mean value theorem) Suppose that $D_a^{k\alpha} f(x) \in C[a,b]$ for k = 0, 1,..., m + 1, where 0 <α< 1, then we have

$$f(b) = \sum_{k=o}^{m} \frac{\left(D_a^{k\alpha} f\right)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha} + \frac{\left(D_a^{(m+1)\alpha} f\right)(\xi)}{\Gamma((m+1)\alpha+1)}(b-1)^{(m+1)\alpha} \quad (3.4)$$

with a ≤ ξ ≤ b, where $D_a^{k\alpha} f$ is sequential Caputo fractional derivative.

Theorem have also been established in [5], here we give another proof by the use of Fractional Cauchy's Mean Value Theorem.

Proof

By the use of fractional Cauchy's mean theorem we can obtain

$$\frac{f(b) - \sum_{k=o}^{m}\frac{\left(D_a^{k\alpha} f\right)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha}}{(b-a)^{(m+1)\alpha}} =$$

$$\frac{(D_a^{\alpha})(\xi) - \sum_{k=1}^{m}\frac{\left(D_a^{k\alpha} f\right)(a)}{\Gamma(k\alpha+1)}(\xi-a)^{k\alpha}}{(m+1)\alpha(\xi-a)^{m\alpha}} = \frac{(D_a^{2\alpha})(\xi_2) - \sum_{k=2}^{m}\frac{\left(D_a^{k\alpha} f\right)(a)}{\Gamma(k\alpha+1)}(\xi-a)^{k\alpha}}{(m+1)\alpha\cdot n\alpha(\xi_2-a)^{(m-1)\alpha}}$$

$$= \ldots = \frac{(D_a^{m\alpha} f)(\xi_m) - \frac{(D_a^{n\alpha} f)(a)}{\Gamma(n\alpha+1)}(\xi_m-a)^{k\alpha}}{(m+1)\alpha\cdot m\alpha\ldots 2\alpha(\xi_m-a)^{\alpha}} = \frac{(D_a^{(m+1)\alpha})(\xi_{m+1})}{(m+1)\alpha\cdot m\alpha\ldots\cdot\alpha}$$

So that we have

$$f(b) = \sum_{k=o}^{m}\frac{\left(D_a^{k\alpha} f\right)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha} + \frac{(D_a^{(m+1)\alpha} f)(\xi_{m+1})}{\Gamma((m+1)\alpha+1)}(b-a)^{(m+1)\alpha}$$

The proof of Theorem is completed.

Theorem

(Generalized Cauchy's mean value theorem) Suppose that $D_a^{k\alpha} f(x), D_a^{k\alpha} g(x) \in C[a,b]$ for k = 0,1, ...m + 1; where 0≤α≤ 1, then we have

$$\frac{f(b)-\sum_{k=o}^{m}\frac{\left(D_a^{k\alpha}f\right)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha}}{g(b)-\sum_{k=o}^{m}\frac{\left(D_a^{k\alpha}g\right)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha}}+\frac{\left(D_a^{(m+1)\alpha}f\right)(\xi)}{\left(D_a^{(m+1)\alpha}g\right)(\xi)} \quad (3.5)$$

Proof

By the use of fractional Cauchy's mean theorem we have

$$\frac{f(b)-\sum_{k=o}^{m}\frac{\left(D_a^{k\alpha}f\right)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha}}{g(b)-\sum_{k=o}^{m}\frac{\left(D_a^{k\alpha}g\right)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha}}=\frac{(D_a^{\alpha}f)(\xi_1)-\sum_{k=1}^{m}\frac{\left(D_a^{k\alpha}f\right)(a)}{\Gamma(k\alpha+1)}(\xi_1-a)^{k\alpha}}{(D_a^{\alpha}g)(\xi_1)-\sum_{k=1}^{m}\frac{\left(D_a^{k\alpha}g\right)(a)}{\Gamma(k\alpha+1)}(\xi_1-a)^{k\alpha}}$$

and

$$\frac{(D_a^{\alpha}f)(\xi_1)-\sum_{k=1}^{m}\frac{\left(D_a^{k\alpha}f\right)(a)}{\Gamma(k\alpha+1)}(\xi_1-a)^{k\alpha}}{(D_a^{\alpha}g)(\xi_1)-\sum_{k=1}^{m}\frac{\left(D_a^{k\alpha}g\right)(a)}{\Gamma(k\alpha+1)}(\xi_1-a)^{k\alpha}}=\frac{(D_a^{2\alpha}f)(\xi_2)-\sum_{k=2}^{m}\frac{\left(D_a^{k\alpha}f\right)(a)}{\Gamma(k\alpha+1)}(\xi-a)^{k\alpha}}{(D_a^{2\alpha}g)(\xi_2)-\sum_{k=2}^{m}\frac{\left(D_a^{k\alpha}g\right)(a)}{\Gamma(k\alpha+1)}(\xi-a)^{k\alpha}}$$

$$=\ldots=\frac{(D_a^{m\alpha}f)(\xi_m)-\frac{(D_a^{m\alpha}f)(a)}{\Gamma(m\alpha+1)}(\xi_m-a)^{k\alpha}}{(D_a^{m\alpha}g)(\xi_m)-\frac{(D_a^{m\alpha}g)(a)}{\Gamma(m\alpha+1)}(\xi_m-a)^{k\alpha}}=\frac{(D_a^{(m+1)\alpha}f)(\xi_{m+1})}{(D_a^{(m+1)\alpha}g)(\xi_{m+1})}$$

The proof of Theorem 3.5 is completed.

Remark

1. Theorem is essentially new.
2. Set $g(x)=\frac{(x-a)^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)}$, then theorem reduces to previous theorem

Generalized Taylor's formula and Cauchy's formula with multivariate

In this section, we discuss Generalized Taylor's formula and Cauchy's formula with multivariate. First, We discuss Generalized Taylor's formula and Cauchy's formula with the Lagrange remainder term.

Theorem

(Generalized Taylor's formula with multivariate) Let D be a convex domain, (x_0, y_0), $(x, y) \in D$, then

$$f(x,y)=\sum_{k=o}^{m}\frac{D_a^{k\alpha}f(x_0,y_0)}{\Gamma(k\alpha+1)}+\frac{D^{(m+1)\alpha}f(\xi,n)}{\Gamma((m+1)\alpha+1)} \quad (4.1)$$

where

$\xi=x_0+\theta(x-x_0)=x_0+\theta\Delta x, \eta=y_0+\theta\Delta y, (0<\theta<1)$

and $D^{k\alpha}f(x_0,y_0), D^{(n+1)\alpha}f(\xi,n)$ are dened in above.

Proof

In previous Theorem replacing function f by φ, and set a = 0, b = 1, then we have

$$\varphi(1)=\sum_{k=0}^{n}\frac{\varphi^{(k\alpha)}(0)}{\Gamma(k\alpha+1)}+\frac{\varphi^{((n+1)\alpha)}(\theta)}{\Gamma((n+1)\alpha+1)},(0<\theta<1) \quad (4.2)$$

On the other hand, set $\varphi(t) = f(x_0 + t(x-x_0), y_0 + t(y-y_0))$, and by Proposition we have

$$\varphi(1)=f(x,y),\varphi^{(k\alpha)}(0)=D^{k\alpha}f(x_0,y_0),\varphi^{((n+1)\alpha)}(\theta)=D^{(n+1)\alpha}f(\xi,\eta) \quad (4.3)$$

Substituting above equation into first equation, then Theorem is completed.

Theorem

(Generalized Cauchy's formula with multivariate) Let D be a convex domain, (x_0, y_0), $(x; y) \in D$, then

$$\frac{f(x,y)-\sum_{k=o}^{m}\frac{D_a^{k\alpha}f(x_0,y_0)}{\Gamma(k\alpha+1)}}{g(x,y)-\sum_{k=o}^{m}\frac{D_a^{k\alpha}g(x_0,y_0)}{\Gamma(k\alpha+1)}}+\frac{D^{(n+1)\alpha}f(\xi,n)}{D^{(n+1)\alpha}g(\xi,n)} \quad (4.4)$$

where $\xi=x_0+\theta(x-x_0)=x_0+\theta\Delta x, \eta=y_0+\theta\Delta y, (0<\theta<1)$

Proof

replacing function f by φ, g by ψ and set a = 0, b = 1, then we get

$$\frac{\varphi(1)-\sum_{k=0}^{n}\frac{\varphi^{(k\alpha)}(0)}{\Gamma(k\alpha+1)}}{\psi(1)-\sum_{k=0}^{n}\frac{\varphi^{(k\alpha)}(0)}{\Gamma(k\alpha+1)}}=\frac{\varphi^{(m+1)\alpha}(\theta)}{\psi^{(m+1)\alpha}(\theta)} \quad (4.5)$$

On the other hand, set

$\varphi(t)=f(x_0+t(x-x_0), y_0+t(y-y_0))$,

$\psi(t)=g(x_0+t(x-x_0), y_0+t(y-y_0))$ by the Proposition we have

$$\varphi(1)=f(x,y),\varphi^{(k\alpha)}(0)=D^{k\alpha}f(x_0,y_0),\varphi^{((n+1)\alpha)}(\theta)=D^{(n+1)\alpha}f(\xi,\eta) \quad (4.6)$$

$$\psi(1)=f(x,y),\psi^{(k\alpha)}(0)=D^{k\alpha}f(x_0,y_0),\psi^{((n+1)\alpha)}(\theta)=D^{(n+1)\alpha}f(\xi,\eta) \quad (4.7)$$

Substituting the above two equation from the previous equation Theorem is completed.

Now, we set $\varphi(t)=f(x_1+t\Delta x_1, x_2+t\Delta x_2, \ldots, x_2+t\Delta x_n)$ where $\Delta x_1=y_1-x_1, \Delta x_2=y_2-x_2, \ldots, \Delta x_n=y_n-x_n$

We can obtain the following Proposition by a process analogous to previous Proposition

Proposition

Let $(x_1,\ldots,xn)$; $(y_1,\ldots, y_n) \in D$, where $D \subset R^n$ is a convex domain, then

$$(D^{\mu}f)(x_1+s\Delta x_1, x_2+s\Delta x_2, \ldots, x_n+s\Delta x_n)=(D^{\mu}\varphi)(s) \quad (4.8)$$

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$$(D^{\mu} f)(y_1, y_2, ..., y_n) = (D^{\mu}\varphi)(1) \quad (4.9)$$

Theorem

(Generalized Taylor's formula with multivariate) Let Suppose that $D_a^{k\alpha} f(x_1, x_2, ..., x_n)$ are continuous in D, for k = 0, 1,...,m + 1; where 0≤α≤ 1, then we have

$$f(y_1, y_2, ..., y_n) = \sum_{k=0}^{m} \frac{D^{k\alpha} f(x_1, x_2, ..., x_n)}{\Gamma(k\alpha+1)} + \frac{D^{(m+1)\alpha} f(\xi_1, \xi_2, ..., \xi_n)}{\Gamma((m+1)\alpha+1)} \quad (4.10)$$

where $\xi_i = x_i + \theta(y_i - x_i), i = 1, 2, ..., n, 0 < \theta < 1$

Theorem

(Generalized Cauchy's formula with multivariate) Let Suppose that $D_a^{k\alpha} f(x_1, x_2, ..., x_n)$, $D_a^{k\alpha} g(x_1, x_2, ..., x_n)$ are continuous in D, for k = 0, 1,...,m+ 1; where 0≤α≤1, then we have

$$\frac{f(y_1, y_2, ..., y_n) - \sum_{k=0}^{m} \frac{D^{k\alpha} f(x_1, x_2, ..., x_n)}{\Gamma(k\alpha+1)}}{g(y_1, y_2, ..., y_n) - \sum_{k=0}^{m} \frac{D^{k\alpha} g(x_1, x_2, ..., x_n)}{\Gamma(k\alpha+1)}} = \frac{D^{(m+1)\alpha} f(\xi_1, \xi_2, ..., \xi_n)}{D^{(m+1)\alpha} g(\xi_1, \xi_2, ..., \xi_n)} \quad (4.11)$$

where $\xi_i = x_i + \theta (y_i - x_i)$; i = 1, 2,...,n.

Next Let us discuss Generalized Cauchy's formula and Cauchy's formula with integral remainder term.

Lemma

Suppose that $\varphi^{(k\alpha)}(t) \in C[0,1]$ for $k = 0, 1, ... m+1$ where 0 <α ≤1, then we have

$$\varphi(t) = \sum_{k=0}^{m} \frac{\varphi^{k\alpha}(0)}{\Gamma(k\alpha+1)} t^{k\alpha} + \frac{1}{\Gamma((m+1)\alpha)} \int_0^t (t-T)^{(m+1)\alpha-1} \varphi^{(m+1)\alpha}(T) dT \quad (4.12)$$

when t = 1, then

$$\varphi(1) = \sum_{k=0}^{m} \frac{\varphi^{k\alpha}(0)}{\Gamma(k\alpha+1)} t^{k\alpha} + \frac{1}{\Gamma((m+1)\alpha)} \int_0^1 (1-T)^{(m+1)\alpha-1} \varphi^{(m+1)\alpha}(T) dT \quad (4.13)$$

Proof

By Laplace Transform, we have

$$L\{\frac{1}{\Gamma((m+1)\alpha)} \int_0^t (t-T)^{(m+1)\alpha-1} \varphi^{(m+1)}(T) dT\}(s)$$

$$= L\{D^{-(m+1)\alpha} D^{(m+1)\alpha} \varphi(t)\}(s)$$

$$= s^{-(m+1)\alpha} L\{D^{(m+1)\alpha} \varphi(t)\}(s)$$

$$= s^{-(m+1)\alpha} [s^{(m+1)\alpha} \hat{\varphi}(s) - \sum_{k=0}^{m} s^{(m-k)\alpha} \varphi^{k\alpha}(0)]$$

$$= \hat{\varphi}(s) - \sum_{k=0}^{m} \frac{\varphi^{k\alpha}(0)}{s^{k\alpha+1}}$$

By inverse Laplace Transform, we obtain

$$\frac{1}{\Gamma((m+1)\alpha)} \int_o^t (t-T)^{(m+1)\alpha-1} \varphi^{(m+1)}(T) dT = \varphi(t) - \sum_{k=0}^{m} \frac{\varphi^{k\alpha}(0)}{\Gamma(k\alpha+1)} t^{k\alpha},$$

and

$$\varphi(1) = \sum_{k=0}^{m} \frac{\varphi^{k\alpha}(0)}{\Gamma(k\alpha+1)} + \frac{1}{\Gamma((m+1)\alpha)} \int_o^t (t-T)^{(m+1)\alpha-1} \varphi^{(m+1)}(T) dT$$

Lemma is completed.

The following Theorem can be directly obtained from Lemma and Proposition.

Theorem

(Generalized Taylor's formula with multivariate) Suppose that $D_a^{k\alpha} f(x_1, x_2, ..., x_n)$ are continuous in D, for k = 0, 1,...,m + 1, where 0 ≤α≤ 1, then we have

$$f(y_1, y_2, ..., y_n) = \sum_{k=0}^{m} \frac{D^{k\alpha} f(x_1, x_2, ..., x_n)}{\Gamma(k\alpha+1)}$$

$$+ \frac{1}{\Gamma((m+1)\alpha)} \int_0^1 (1-t)^{(m+1)\alpha-1} D^{(m+1)\alpha} f(x_1 + s(y_1 - x_1), x_2$$

$$+ s(y_2 - x_2), ..., x_n + s(y_n - x_n)) ds \quad (4.14)$$

Theorem

(Generalized Cauchy's formula with multivariate) Let Suppose that $D_a^{k\alpha} f(x_1, x_2, ..., x_n)$, $D_a^{k\alpha} g(x_1, x_2, ..., x_n)$ are continuous in D, for k = 0; 1,...,m+ 1, where 0 <α≤1, then we have

$$\frac{f(y_1, y_2, ..., y_n) - \sum_{k=0}^{m} \frac{D^{k\alpha} f(x_1, x_2, ..., x_n)}{\Gamma(k\alpha+1)}}{g(y_1, y_2, ..., y_n) - \sum_{k=0}^{m} \frac{D^{k\alpha} g(x_1, x_2, ..., x_n)}{\Gamma(k\alpha+1)}}$$

$$= \frac{\int_0^1 (1-t)^{(m+1)\alpha-1} D^{(m+1)\alpha} f(x_1 + s(y_1 - x_1), x_2 + s(y_2 - x_2), ..., x_n + s(y_n - x_n)) ds}{\int_0^1 (1-t)^{(m+1)\alpha-1} D^{(m+1)\alpha} g(x_1 + s(y_1 - x_1), x_2 + s(y_2 - x_2), ..., x_n + s(y_n - x_n)) ds} \quad (4.15)$$

Remark

Last, Let us consider some special cases

When n = 0, 0 < α < 1, then we get

$$f(y_1) = \sum_{k=0}^{m} \frac{D^{k\alpha} f(x_1, x_2, ..., x_n)}{\Gamma(k\alpha+1)} + \frac{1}{\Gamma((m+1)\alpha)} \int_0^1 (1-t)^{(m+1)\alpha-1} D^{(m+1)\alpha} f(x_1 + s(y_1 - x_1)) dt \quad (4.16)$$

Now we have

$$(D^{-\nu} f)(y) = \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} f(x_1 + t(y - x_1)) dt, \quad (\nu \in R^+)$$

it is easy to verify that

$$(D^{-\nu} f)(y) = (y - x_1)^{-\nu} \frac{1}{\Gamma(\nu)} \int_{x_1}^{y} (y-T)^{\nu-1} f(T) dt = (y - x_1)^{-\nu} [D_{x_1}^{-\nu}](y)$$

where $\left[D_{x_1}^{-\nu} f\right](y)$ is Riemann-Liouville integral.

Similarly we can obtain

$$(D^{\mu} f)(y) = (y - x_1)^{\mu} [D_{x_1}^{\mu} f](y) \quad (4.17)$$

where $\left[D_{x_1}^{-\mu} f\right](y)$ is Caputo fractional derivative.

Therefore, combining formula (4.16) with (4.17), we get

$$f(y_1)=\sum_{k=0}^{m}\frac{(y_1-x_1)^{k\alpha}}{\Gamma(k\alpha+1)}[D_{x_1}^{k\alpha}f](x_1)+$$
$$\frac{1}{\Gamma((m+1)\alpha)}\int_{x_1}^{y_1}(y_{-1}-T)^{(m+1)\alpha-1}D_{x_1}^{(m+1)\alpha}f(T)dT \quad (4.18)$$

which is the same as previous one

When n = 0; = 1, the generalized Taylor's formula reduced to the classical Taylor's formula

$$f(y_1)=\sum_{k=0}^{m}\frac{(y_1-x_1)^{k}}{k!}f^{k}(x_1)+\frac{1}{m!}\int_{x_1}^{y_1}(y_1-T)^{m}f^{(m+1)}(T)dT$$

Further, Let m = 0, it reduced to the well-known Newton-Leibnitz's fundamental theorem of calculus $f(y_1)=f(x_1)+\int_{x_1}^{y_1}f'(T)dT$

When n > 1, α = 1 we have

$$f(y_1,y_2,...,y_n)=\sum_{k=0}^{m}\frac{1}{k!}(\Delta x_1\frac{\partial}{\partial x_1}+\Delta x_2\frac{\partial}{\partial x_2}+...+\Delta x_n\frac{\partial}{\partial x_n})^k f(x_1,x_2,...,x_n)+$$
$$\frac{1}{m!}\int_0^1(1-t)^m D^{m+1}f(x_1+t\Delta x_1,x_2+t\Delta x_2,...,x_n+t\Delta x_n)dt$$

which is the classical Taylor's formula with multivariate.

Let α = 1, m = 0 in above Theorem then we can get

$$f(y_1,y_2,...,y_n)=f(x_1,x_2,...,x_n)+$$
$$\int_0^1(\Delta x_1\frac{\partial}{\partial x_1}+\Delta x_2\frac{\partial}{\partial x_2}+...+\Delta x_n\frac{\partial}{\partial x_n})f(x_1+t\Delta x_1,x_2+t\Delta x_2,...,x_n+t\Delta x_n)dt$$

Conflict of interest

No authors have a conflict of interest or any financial tie to disclose.

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