On Fractional Taylor’s Formula and Fractional Cauchy’ Formula with Multivariate

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Abstract

In this paper, a generalized fractional Taylor’s formula and Cauchy’s formula of the kind

\[ f(x) = \sum_{n=0}^{\infty} \frac{D^{\alpha n}f(x)}{\Gamma(\alpha n + 1)} (x-a)^{\alpha n} + R_{\alpha n}(x,a) \]  

(1.4)

where \( 0 < \alpha_n \leq 1, n = 1, 2, \ldots \) and \( D^{\alpha n}f \) is the sequential fractional Riemann-Liouville derivative of order \( \alpha n \).

On the other hand, a variant of the generalized Taylor’s series was given by Dzherbashyan and Nersesyan [3]. For \( f \) having all of the required continuous derivatives, they obtained

\[ f(x) = \sum_{n=0}^{\infty} \frac{D^{\alpha n}f(x)}{\Gamma(\alpha n + 1)} (x-a)^{\alpha n} + R_{\alpha n}(x,a) \]  

(1.5)

where \( \alpha_n \) is the Mittage-Leer function defined by the expression

\[ \alpha_n(x) = \frac{\Gamma(\alpha_n + 1)}{\Gamma(\alpha_n + \alpha_n + 1)} \]

and the sequential fractional Riemann-Liouville derivative is denoted by

\[ D^{\alpha n}_a f = D^{\alpha_1}_a \cdot D^{\alpha_2}_a \cdot \ldots \cdot D^{\alpha_n}_a f \]

(\( n \)-times)

Recently, Zaid M. Odibat, Nabil T. Shawagfeh [5] obtained a new generalized Taylor’s formula of this kind

\[ f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^{\alpha n}}{\Gamma(\alpha n + 1)} (D^{\alpha n}f(a)) + \frac{1}{\Gamma(\alpha n + 1)} \int_0^x (x-t)^{\alpha n-1} (D^{\alpha n}f(t)) dt \]  

(1.5)

where \( \alpha_n \) is the Mittage-Leer function defined by the expression

\[ \alpha_n(x) = \frac{\Gamma(\alpha_n + 1)}{\Gamma(\alpha_n + \alpha_n + 1)} \]

On the other hand, a variant of the generalized Taylor’s series was given by Dzherbashyan and Nersesyan [3]. For \( f \) having all of the required continuous derivatives, they obtained

\[ f(x) = \sum_{n=0}^{\infty} \frac{D^{\alpha n}f(x)}{\Gamma(\alpha n + 1)} (x-a)^{\alpha n} + R_{\alpha n}(x,a) \]  

(1.6)

Introduction

The ordinary Taylor’s formula has been generalized by many authors. Riemann [1] had already written a formal version of the generalized Taylor’s series:

\[ f(x + h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} h^n \]  

(1.1)

where \( J^{\alpha n} \) is the Riemann-Liouville fractional integral of order \( \alpha + n \). Afterward, Watanabe [2] obtained the following relation:

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(\alpha n + 1)} (x-a)^{\alpha n} + R_{\alpha n}(x,a) \]  

(1.2)

where \( D^{\alpha n}_a f \) is the Riemann-Liouville fractional derivative of order \( \alpha n \).

On the other hand, a variant of the generalized Taylor’s series was given by Dzherbashyan and Nersesyan [3]. For \( f \) having all of the required continuous derivatives, they obtained

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x-a)}{\Gamma(\alpha n + 1)} (x-a)^{\alpha n} + R_{\alpha n}(x,a) \]  

(1.3)

where \( 0 < \alpha_n \leq 1, n = 1, 2, \ldots \) and \( D^{\alpha n}_{(x-a)} f \) is the sequential fractional Riemann-Liouville derivative of order \( \alpha n \).

Under certain conditions for \( f \) and \( \alpha \in [0, 1] \), Trujillo et al. [4] introduce the following generalized Taylor’s formula:

\[ f(x) = \sum_{j=0}^{n} \frac{c_j (x-a)^j}{\Gamma(j+1)} + R_{\alpha n}(x,a) \]  

(1.4)

where \( c_j = \frac{(D^{\alpha n}_{(x-a)} f)(a)}{\Gamma(\alpha n + 1)} \) and \( \alpha_n \) is the Mittage-Leer function defined by the expression

\[ \alpha_n(x) = \frac{\Gamma(\alpha_n + 1)}{\Gamma(\alpha_n + \alpha_n + 1)} \]
In this paper, we will give a new definition of fractional integral and derivative of function with multivariate, derive fractional Taylor’s formula with multivariate (a) with the Lagrange remainder term; (b) with integral remainder term. and derive fractional Cauchy formula with multivariate (a) with the Lagrange remainder term; (b) with integral remainder term.

As far as we are aware, this definition and results have not been published elsewhere previously.

**Definitions and Properties**

For the concept of fractional derivative we will adopt Caputo’s definition which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which in the case in most physical processes. More detailed information on fractional calculus may be found in these books [7-14].

**Definition**

A function \(f(x)(x > 0)\) is said to be in the space \(C_\alpha\) if it can be written as \(f(x) = x^\alpha f_1(x)\) for \(p > 0\) where \(f_1(x)\) is continuous in \((0,\infty)\) and it is said to be in the space \(C_\alpha\) if \(f(x) \in C_\alpha\), \(m \in N\).

**Definition**

\(\{7-14\}\) Let \(f(x) \in C_\alpha(x, \infty)\), the Riemann-Liouville integral operator of order \(\alpha > 0\) is defined as

\[
\left(D^\alpha_x\right) f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-x)^{\alpha-1} f(t)dt, x > a
\]

**Definition**

\(\{7-14\}\) Let \(f(x) \in C_\alpha^{(m)}(a, \infty)\), the Caputo fractional derivative of \(f(x)\) of order \(\alpha > 0\) is defined as

\[
\left(D^\alpha_x\right) f(x) = \left(D^\alpha_x\right)^{(m)} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (t-x)^{m-\alpha-1} f^{(m)}(t)dt
\]

for \(-1 < \alpha \leq m, m \in N, x \geq a\)

**Definition**

\(\{7-14\}\) Let \(D^\alpha\) be Riemann-Liouville integral operator of order \(\alpha > 0\), be Caputo fractional derivative operator of order \(\alpha > 0, 0 < \alpha \leq 1\), \(D^\alpha f(x) \in C(a, b)\) then

\[
\left[D^\alpha D^\alpha_x\right] f(x) = f(x) - f(a)
\]

In order to derive fractional Taylor’s formula and Cauchy formula of a function with multivariate, we will get the following integrate denition of a function \(f(x; y)\), \((x; y) \in D\), where \(D\) is a convex domain.

**Definition**

Let \((x_0, y_0, h(x, y)) \in D, \Delta x = x - x_0, \Delta y = y - y_0, 0 \leq s \leq 1\) then

\[
(D^\alpha f)(x_0 + s\Delta x, y_0 + s\Delta y) = \int_0^1 f \left( x_0 + t\Delta x, y_0 + t\Delta y \right) dt
\]

(2.1)

when \(s = 1\), define

\[
(D^\alpha f)(x, y) = \int_0^1 f \left( x_0 + t\Delta x, y_0 + t\Delta y \right) dt
\]

(2.2)

**Proposition**

Let \(k \in N, (x_0, y_0) \in D, \Delta x = x - x_0, \Delta y = y - y_0, 0 \leq s \leq 1\) then

\[
(D^{-\alpha} f)(x_0 + s\Delta x, y_0 + s\Delta y) = \frac{1}{\Gamma(\alpha)} \int_0^1 (s-t)^{\alpha-1} f \left( x_0 + t\Delta x, y_0 + t\Delta y \right) dt
\]

(2.3)

**Proof**

By Definition we have

\[
(D^{-\alpha} f)(x_0 + s\Delta x, y_0 + s\Delta y) = \int_0^1 D f \left( x_0 + t\Delta x, y_0 + t\Delta y \right) dt
\]

\[
= \int_0^1 dt \int_0^t f \left( x_0 + u\Delta x, y_0 + u\Delta y \right) du
\]

\[
= \int_0^1 du \int_0^u f \left( x_0 + u\Delta x, y_0 + u\Delta y \right) du = \int_0^1 (s-u) f \left( x_0 + u\Delta x, y_0 + u\Delta y \right) du
\]

By induction, it is not hard to prove that

\[
(D^{-\alpha} f)(x_0 + s\Delta x, y_0 + s\Delta y) = \frac{1}{\Gamma(\alpha)} \int_0^1 (s-t)^{\alpha-1} f \left( x_0 + t\Delta x, y_0 + t\Delta y \right) dt
\]

(2.4)

Now we can define fractional integral of \(f(x; y)\) of order \(\gamma\)

**Definition**

\(\gamma \in R^+, (x_0, y_0) \in D, \Delta x = x - x_0, \Delta y = y - y_0, 0 \leq s \leq 1\) define

\[
(D^{-\gamma} f)(x_0 + s\Delta x, y_0 + s\Delta y) = \frac{1}{\Gamma(\gamma)} \int_0^1 (s-t)^{\gamma-1} f \left( x_0 + t\Delta x, y_0 + t\Delta y \right) dt
\]

(2.5)

For convenience, Let us set

\[
\varphi(t) = f \left( x_0 + t\Delta x, y_0 + t\Delta y \right) \text{ then we have}
\]

**Proposition**

\[
(D^{-\gamma} f)(x_0 + s\Delta x, y_0 + s\Delta y) = \frac{1}{\Gamma(\gamma)} \int_0^1 (s-t)^{\gamma-1} f \left( x_0 + t\Delta x, y_0 + t\Delta y \right) dt
\]

(2.6)

and

\[
(D^{-\gamma} f)(x, y) = \frac{1}{\Gamma(\gamma)} \int_0^1 (1-t)^{\gamma-1} f \left( x_0 + t\Delta x, y_0 + t\Delta y \right) dt
\]

(2.7)

By Proposition it is easy to see that

**Proposition**

Let \((x_0, y_0, h(x, y)) \in D, \Delta x = x - x_0, \Delta y = y - y_0, 0 \leq s \leq 1\) then

\[
(D^{-\gamma} f)(x_0 + s\Delta x, y_0 + s\Delta y) = \frac{1}{\Gamma(\gamma)} \int_0^1 (s-t)^{\gamma-1} f \left( x_0 + t\Delta x, y_0 + t\Delta y \right) dt
\]

(2.8)

Let \(a, \beta \in R^+\) then

\[
D^{-\alpha} \frac{\partial}{\partial x} f(x_0 + s\Delta x, y_0 + s\Delta y) = D^{-\alpha+\beta} f(x_0 + s\Delta x, y_0 + s\Delta y)
\]

and

\[
D^{-\alpha} \frac{\partial}{\partial x} f(x, y) = D^{-\alpha+\beta} f(x, y)
\]

**Definition**

If \(n \in N\) define

\[
D^n f(x_0 + s\Delta x, y_0 + s\Delta y) = \left(\Delta x^\alpha \frac{\partial}{\partial x}\right)^n f(x_0 + s\Delta x, y_0 + s\Delta y)
\]

(2.9)
\[ D^\alpha f(x, y) = (\Delta_x D^\alpha + \Delta_y D^\alpha) f(x, y) \tag{2.9} \]

**Definition**

Let \( \mu > 0 \) and let \( n \) be the smallest integer exceeding \( \mu \), we define the fractional derivative of \( f \) of order \( \mu \) as following

\[ D^\mu f \left( x, y + s \Delta x, y + s \Delta y \right) = (D^{\mu-n} D^n f)(x, y + s \Delta x, y + s \Delta y) \tag{2.10} \]

and

\[ D^n f(x, y) = (D^n \varphi)(y) \tag{2.11} \]

**Proposition**

Let \( \varphi(s) = f(x + s \Delta x, y + s \Delta y) \) then

\[ D^n f(x, y) = \varphi^{(n)}(y) \tag{2.12} \]

and

\[ D^n f(x, y) = \varphi^{(n)}(1) \tag{2.13} \]

**Proposition**

Let \( \mu \in \mathbb{R}^+ \) then

\[ (D^\mu f)(x, y) = (D^\mu \varphi)(s) \tag{2.14} \]

and

\[ (D^\mu f)(x, y) = (D^\mu \varphi)(1) \tag{2.15} \]

**Generalized Taylor’s mean value theorem and generalized Cauchy’s mean value theorem with one variable**

In this section, we will give fractional Taylor’s mean value theorem and Cauchy’s mean value theorem involving the sequential Caputo fractional derivative with one variable. Let us begin with some basic fractional mean value theorems.

**Theorem**

(Fractional Lagrange’s mean value theorem) Suppose that \( f(x) \in \mathbb{C}[a, b] \) and \( D^\alpha f(x) \in \mathbb{C}[a, b] \), for \( 0 \leq \alpha \leq 1 \), then we have

\[ f(b) - f(a) = \frac{1}{\Gamma(\alpha + 1)} D^\alpha f(\xi)(b - a)^\alpha \tag{3.1} \]

with \( a \leq \xi \leq b \)

**Proof**

In view of Proposition we have

\[ f(b) - f(a) = \left[ D^\alpha D^\alpha f \right](b) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b \left( b - T \right)^{\alpha-1} D^\alpha f(T) dT \]

\[ = \left[ D^\alpha f \right](\xi) \frac{1}{\Gamma(\alpha + 1)} \int_a^b \left( b - T \right)^{\alpha-1} dT = \left[ D^\alpha f \right](\xi) \frac{(b - a)^\alpha}{\Gamma(\alpha + 1)} \]

From above Theorem it is easy to obtain

**Corollary**

(Fractional Rolle’s mean value theorem) Suppose that \( f(x) \in \mathbb{C}[a, b], D^\alpha f(x), D^\alpha g(x) \in \mathbb{C}[a, b] \) for \( 0 \leq \alpha \leq 1 \), and \( f(a) = f(b) \), then there exists \( \xi \in (a, b) \) such that

\[ D^\alpha f(\xi) = 0 \tag{3.2} \]

Now we can derive fractional Cauchy’s mean value theorem with one variable.

**Theorem**

(Fractional Cauchy’s mean value theorem) Suppose that \( f(x), g(x) \in \mathbb{C}[a, b] \) and \( D^\alpha f(x), D^\alpha g(x) \in \mathbb{C}[a, b] \), for \( 0 \leq \alpha \leq 1 \), then we have

\[ \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{D^\alpha f(\xi)}{D^\alpha g(\xi)} \tag{3.3} \]

with \( a \leq \xi \leq b \)

**Proof**

Set \( F(x) = \left[ f(b) - f(a) \right] \left[ g(x) - g(a) \right] - \left[ f(x) - f(a) \right] \left[ g(b) - g(a) \right] \), then \( F(a) = F(b) = 0 \) in view of fractional Rolle’s mean value Corollary so that there exists \( \xi \in (a, b) \), such that

\[ D^\alpha F(\xi) = 0 \]

Therefore, we have

\[ \left[ f(b) - f(a) \right] D^\alpha g(\xi) - D^\alpha f(\xi) \left[ g(b) - g(a) \right] = 0 \]

Theorem is completed.

**Theorem**

(Fractional Taylor’s mean value theorem) Suppose that \( D^\alpha f(x) \in \mathbb{C}[a, b] \) for \( k = 0, 1, \ldots, m + 1 \), where \( 0 < \alpha < 1 \), then we have

\[ f(b) - f(a) = \sum_{\nu = 0}^m \frac{D^{\nu} f(a)}{\Gamma(\nu + 1)} (b - a)^\nu + \frac{D^{m+1\alpha} f(\xi)}{\Gamma(m + 1)} (b - 1)^{m+1\alpha} \tag{3.4} \]

with \( a \leq \xi \leq b \), where \( D^{m+1\alpha} f \) is sequential Caputo fractional derivative.

Theorem have also been established in [5], here we give another proof by the use of fractional Cauchy’s Mean Value Theorem.

**Proof**

By the use of fractional Cauchy’s mean theorem we can obtain

\[ f(b) = \sum_{\nu = 0}^m \frac{D^{\nu} f(a)}{\Gamma(\nu + 1)} (b - a)^\nu \]

\[ = \frac{D^{m+1\alpha} f(\xi)}{\Gamma(m + 1)} (b - a)^{m+1\alpha} \]

\[ \text{So that we have} \]

\[ f(b) = \sum_{\nu = 0}^m \frac{D^{\nu} f(a)}{\Gamma(\nu + 1)} (b - a)^\nu + \frac{D^{m+1\alpha} f(\xi)}{\Gamma(m + 1)} (b - a)^{m+1\alpha} \]

The proof of Theorem is completed.
\[ f(b) \sim \sum_{\kappa=0}^{n} \left( D_{\kappa}^{(\alpha)} f(a) (b-a)^{\alpha \kappa} \right) + \left( D_{\kappa}^{(n+1)\alpha} f (\xi) \right) \]  
\[ g(b) \sim \sum_{\kappa=0}^{n} \left( D_{\kappa}^{(\alpha)} g(a) (b-a)^{\alpha \kappa} \right) + \left( D_{\kappa}^{(n+1)\alpha} g (\xi) \right) \]  
(3.5)

**Proof**

By the use of fractional Cauchy's mean theorem we have

\[ f(b) - f(a) = \sum_{\kappa=0}^{n} \frac{D_{\kappa}^{(\alpha)} f(a)}{\Gamma(k\alpha+1)} (b-a)^{\alpha \kappa} + \frac{D_{\kappa}^{(n+1)\alpha} f (\xi)}{\Gamma((n+1)\alpha+1)} \]

and

\[ g(b) - g(a) = \sum_{\kappa=0}^{n} \frac{D_{\kappa}^{(\alpha)} g(a)}{\Gamma(k\alpha+1)} (b-a)^{\alpha \kappa} + \frac{D_{\kappa}^{(n+1)\alpha} g (\xi)}{\Gamma((n+1)\alpha+1)} \]

**Remark**

1. Theorem is essentially new.
2. Set \( g(x) = \frac{x-a}{(n+1)\alpha} \), then theorem reduces to previous theorem.

**Generalized Taylor’s formula and Cauchy’s formula with multivariate**

In this section, we discuss Generalized Taylor’s formula and Cauchy’s formula with multivariate. First, we discuss Generalized Taylor’s formula and Cauchy’s formula with the Lagrange remainder term.

**Theorem**

(Generalized Taylor’s formula with multivariate) Let \( D \) be a convex domain, \((x_0,y_0), (x,y) \in D\), then

\[ f(x,y) = \sum_{\kappa=0}^{n} \frac{D_{\kappa}^{(\alpha)} f(x_0,y_0)}{\Gamma(k\alpha+1)} (x-x_0)^{\alpha \kappa} + \frac{D_{\kappa}^{(n+1)\alpha} f (x_0, y_0, \xi, n)}{\Gamma((n+1)\alpha+1)} \]  
(4.1)

where

\[ \xi = x_0 + \theta(x - x_0) = x_0 + \theta \Delta x, \eta = y_0 + \theta \Delta y, (0 < \theta < 1) \]

and \( D_{\kappa}^{(\alpha)} f (x_0, y_0), D_{\kappa}^{(n+1)\alpha} f (\xi, n) \) are dened in above.

**Proof**

In previous Theorem replacing function \( f \) by \( \varphi \), and set \( a = 0, b = 1 \), then we have

\[ \varphi(1) = \sum_{\kappa=0}^{n} \frac{\varphi^{(\alpha)}(0)}{\Gamma((n+1)\alpha+1)} + \frac{\varphi^{(n+1)\alpha}(\theta)}{\Gamma((n+1)\alpha+1)}, (0 < \theta < 1) \]  
(4.2)

On the other hand, set \( \varphi(t) = f(x_0 + t(x-x_0), y_0 + t(y-y_0)) \), and by Proposition we have

\[ \varphi(1) = f(x_0 + t(x-x_0), y_0 + t(y-y_0)) \]  
(4.3)

Substituting above equation into first equation, then Theorem is completed.

**Theorem**

(Generalized Cauchy’s formula with multivariate) Let \( D \) be a convex domain, \((x_0,y_0), (x,y) \in D\), then

\[ f(x,y) = \sum_{\kappa=0}^{n} \frac{D_{\kappa}^{(\alpha)} f(x_0,y_0)}{\Gamma(k\alpha+1)} (x-x_0)^{\alpha \kappa} + \frac{D_{\kappa}^{(n+1)\alpha} f (x_0, y_0, \xi, n)}{\Gamma((n+1)\alpha+1)} \]  
(4.4)

\[ g(x,y) = \sum_{\kappa=0}^{n} \frac{D_{\kappa}^{(\alpha)} g(x_0,y_0)}{\Gamma(k\alpha+1)} (y-y_0)^{\alpha \kappa} + \frac{D_{\kappa}^{(n+1)\alpha} g (x_0, y_0, \xi, n)}{\Gamma((n+1)\alpha+1)} \]

where

\[ \xi = x_0 + \theta(x - x_0) = x_0 + \theta \Delta x, \eta = y_0 + \theta \Delta y, (0 < \theta < 1) \]

**Proof**

replacing function \( f \) by \( \varphi \), \( g \) by \( \psi \) and set \( a = 0, b = 1 \), then we get

\[ \psi(1) - \varphi(1) = \sum_{\kappa=0}^{n} \frac{\varphi^{(\alpha)}(0)}{\Gamma((n+1)\alpha+1)} + \frac{\varphi^{(n+1)\alpha}(\theta)}{\Gamma((n+1)\alpha+1)} \]  
(4.5)

On the other hand, set

\[ \varphi(t) = f(x_0 + t(x-x_0), y_0 + t(y-y_0)), \]

\[ \psi(t) = g(x_0 + t(x-x_0), y_0 + t(y-y_0)) \]  
by the Proposition we have

\[ \varphi(1) - \psi(1) = \int_{0}^{1} \varphi^{(\alpha)}(t) dt + \int_{0}^{1} \psi^{(n+1)\alpha}(t) dt = \int_{0}^{1} \int_{0}^{1} \partial^{(\alpha)} f (x_0, y_0, \xi, n) \]  
(4.6)

\[ \psi(1) - \varphi(1) = \int_{0}^{1} \varphi^{(\alpha)}(t) dt + \int_{0}^{1} \psi^{(n+1)\alpha}(t) dt = \int_{0}^{1} \int_{0}^{1} \partial^{(\alpha)} f (x_0, y_0, \xi, n) \]  
(4.7)

Substituting the above two equations from the previous equation Theorem is completed.

Now, we set \( \varphi(t) = f(x_0 + t(x-x_0), x_0 + t(x-x_0), x_0 + t(x-x_0) + t\Delta x_3, \ldots, x_0 + t\Delta x_n) \)

where \( \Delta x_1 = y_1 - x_1, \Delta x_2 = y_2 - x_2, \ldots, \Delta x_n = y_n - x_n \)

We can obtain the following Proposition by a process analogous to previous Proposition.

**Proposition**

Let \((x_1,\ldots,x_n); (y_1,\ldots,y_n) \in D\), where \( D \subset \mathbb{R}^n \) is a convex domain, then

\[ (D^{(\alpha)} f) (x_1 + s\Delta x_1, x_2 + s\Delta x_2, \ldots, x_n + s\Delta x_n) = (D^{(\alpha)} f)(s) \]  
(4.8)
(D^\alpha f)(y_1,y_2,...,y_n) = (D^\alpha \varphi)(1) \hspace{1cm} (4.9)

**Theorem**

(Generalized Taylor’s formula with multivariate) Let Suppose that $D^\alpha f(x_1,x_2,...,x_n)$ are continuous in D, for $k = 0, 1,...,m+1$; where $0 \leq \alpha \leq 1$, then we have

$$f(y_1,y_2,...,y_n) = \sum_{k=0}^{m} \frac{D^\alpha f(x_1,x_2,...,x_n)}{\Gamma((m+1)\alpha+1)} \int_{0}^{1} (1-t)^{(m+1)\alpha} (T) dT$$

where $\xi = x + \theta (y_i - x_i); i = 1, 2,...,n.$

Next Let us discuss Generalized Cauchy's formula and Cauchy's formula with integral remainder term.

**Lemma**

Suppose that $\varphi^{(k\alpha)}(t) \in C[0,1]$ for $k = 0, 1,...,m+1$; where $0 \leq \alpha \leq 1$, then we have

$$\varphi(t) = \sum_{k=0}^{m} \frac{\varphi^{(k\alpha)}(0)}{\Gamma(k\alpha+1)} + \frac{1}{\Gamma((m+1)\alpha+1)} \int_{0}^{1} (t-T)^{(m+1)\alpha} \varphi^{(m+1\alpha)}(T) dT \hspace{1cm} (4.12)$$

when $t = 1$, then

$$\varphi(1) = \sum_{k=0}^{m} \frac{\varphi^{(k\alpha)}(0)}{\Gamma(k\alpha+1)} + \frac{1}{\Gamma((m+1)\alpha+1)} \int_{0}^{1} (1-T)^{(m+1)\alpha} \varphi^{(m+1\alpha)}(T) dT \hspace{1cm} (4.13)$$

**Proof**

By Laplace Transform, we have

$$\frac{1}{\Gamma((m+1)\alpha)} \int_{0}^{s} (t-T)^{(m+1)\alpha} \varphi^{(m+1\alpha)}(T) dT$$

$= L[D^{(m+1)\alpha} \varphi^{m+1\alpha}](s)$

$= s^{-m-1} \int_{0}^{s} d^{m+1\alpha} \varphi^{(m+1\alpha)}(s)$

$= s^{-m-1} [s^{m+1\alpha} \varphi^{(m+1\alpha)}(s)]$

$= \varphi(s) - \sum_{k=0}^{m} \frac{\varphi^{(k\alpha)}(0)}{\Gamma(k\alpha+1)}$

By inverse Laplace Transform, we obtain

$$\frac{1}{\Gamma((m+1)\alpha)} \int_{0}^{s} (t-T)^{(m+1)\alpha} \varphi^{(m+1\alpha)}(T) dT = \varphi(t) - \sum_{k=0}^{m} \frac{\varphi^{(k\alpha)}(0)}{\Gamma(k\alpha+1)}$$

and

$$\varphi(1) = \sum_{k=0}^{m} \frac{\varphi^{(k\alpha)}(0)}{\Gamma(k\alpha+1)} + \frac{1}{\Gamma((m+1)\alpha)} \int_{0}^{1} (t-T)^{(m+1)\alpha} \varphi^{(m+1\alpha)}(T) dT$$

Lemma is completed.

The following Theorem can be directly obtained from Lemma and Proposition.

**Theorem**

(Generalized Taylor’s formula with multivariate) Suppose that $D^\alpha f(x_1,x_2,...,x_n)$ are continuous in D, for $k = 0, 1,...,m+1$; where $0 \leq \alpha \leq 1$, then we have

$$f(y_1,y_2,...,y_n) = \sum_{k=0}^{m} \frac{D^\alpha f(x_1,x_2,...,x_n)}{\Gamma((m+1)\alpha+1)} \int_{0}^{1} (1-t)^{(m+1)\alpha} f(x_1+s(y_i-x_i),x_2$$

$+s(y_j-x_j),...,x_n+s(y_n-x_n)) ds \hspace{1cm} (4.14)$

**Theorem**

(Generalized Cauchy’s formula with multivariate) Let Suppose that $D^\alpha f(x_1,x_2,...,x_n)$ and $D^\alpha g(x_1,x_2,...,x_n)$ are continuous in D, for $k = 0, 1,...,m+1$, where $0 < \alpha < 1$, then we have

$$f(y_1,y_2,...,y_n) = \sum_{k=0}^{m} \frac{D^\alpha f(x_1,x_2,...,x_n)}{\Gamma((m+1)\alpha+1)} \int_{0}^{1} (1-t)^{(m+1)\alpha} f(x_1+s(y_i-x_i),x_2)$$

$$+s(y_j-x_j),...,x_n+s(y_n-x_n)) ds \hspace{1cm} (4.15)$$

**Remark**

Last, Let us consider some special cases

When $n=0, 0 < \alpha < 1$, then we get

$$f(y_1) = \sum_{k=0}^{m} \frac{D^\alpha f(x_1,x_2,...,x_n)}{\Gamma(k\alpha+1)} +$$

$$\frac{1}{\Gamma((m+1)\alpha+1)} \int_{0}^{1} (1-t)^{(m+1)\alpha} f(x_1+s(y_i-x_i)) dt$$

Now we have

$$(D^\alpha)^n f(y)(y) = \frac{1}{\Gamma(n+1)} \int_{0}^{1} (1-t)^{(n+1)\alpha} f(x_1+t(y-x_i)) dt, \hspace{1cm} (\nu \in R^+)$$

it is easy to verify that

$$(D^\alpha)^n f(y) = (y-x_i)^{\nu} \frac{1}{\Gamma(n+1)} \int_{0}^{1} (1-t)^{(n+1)\alpha} f(T) dT = (y-x_i)^{\nu}(D^\alpha)^n f(y)$$

where $[(D^\alpha)^n f(y)](y)$ is Riemann-Liouville integral.

Similarly we can obtain

$$(D^\alpha)^n f(y) = (y-x_i)^{\nu} (D^\alpha)^n f(y) \hspace{1cm} (4.17)$$

where $[(D^\alpha)^n f(y)](y)$ is Caputo fractional derivative.

Therefore, combining formula (4.16) with (4.17), we get
when \( n = 0; \alpha = 1 \), the generalized Taylor’s formula reduced to the classical Taylor’s formula

Further, Let \( m = 0 \), it reduced to the well-known Newton-Leibnitz’s fundamental theorem of calculus \( f(x_i) = f(x_i) + \int_{x_i}^{T} f'(T)dT \)

When \( n > 1, \alpha = 1 \) we have

which is the same as previous one

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which is the classical Taylor’s formula with multivariate.


Conflict of interest

No authors have a conflict of interest or any financial tie to disclose.

References