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A Comparison of Estimation Methods for Generalized Gamma Distribution with One-shot Device Testing Data

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Abstract

Generalized gamma distribution includes many useful lifetime distributions for analyzing lifetime data in reliability and survival studies. In the context of one-shot device testing, it is dicult to collect sucient lifetime information on the one-shot devices, due to the destructive nature of one-shot devices that either left- or right-censored data are collected. In modern life-tests, test devices are subjected to conditions in excess of its normal operation condition in order to induce more failures within a relatively short period of time. Such life-tests are called accelerated life-tests and commonly used for collecting lifetime data. In this paper, we discuss the analysis of one-shot device testing data under accelerated life-tests based on generalized gamma distributions. Both maximum likelihood and least-squares approaches are developed to find the estimates of the model parameters. Furthermore, the estimation on the reliability at a specic mission time as well as on the mean lifetime of the devices are also developed. Both approaches are then compared through comprehensive simulation studies. The results show that both approaches are quite satisfactory in terms of biases, root mean square errors, and numbers of cases of convergence. In general, the maximum likelihood approach is comparably stable to nd the estimates.

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Accelerated life-tests, One-shot devices, Generalized gamma distribution, Fisher scoring, Least-squares.

Introduction

A unit that performs its function only once, and cannot be used for testing more than once is called one-shot device. In life testing of one-shot device, for each test unit, only the condition (success/failure) at an inspection time can be observed. No exact failure times can be obtained from the test. As a result, the lifetime of the test unit is either right-censored (success) or left-censored (failure). For instance, Fan, et al. [1] considered electro-explosive devices that are detonated by inducing a current to excite inner powder. Those devices cannot be used any further after detonation, regardless of whether the detonation is successful or not. Moreover, Morris [2] analyzed battery data from destructive life-tests. The lifetimes of those units cannot be obtained from the tests.

Accelerated life-tests wherein test units are subjected to elevated stress levels, are usually performed to induce quick failures and to collect sucient failure information about the devices. Then, a lifetime distribution model is used to extrapolate from the failure data collected at elevated stress levels to the lifetime distribution under the normal operating condition. In this regard, reliability analysis for one-shot device testing data from constant stress accelerated life-tests has recently received increasing attention. Interested readers may refer to [1-10].

In practice, lifetime data analysis in reliability and survival studies are very often done based on model assumption. Choosing the besttting distribution for a given data set is an important issue, because the eect due to model mis-specication can be severe. Ling and Balakrishnan [11] conducted model mis-specication analyses of Weibull and gamma models based on one-shot device test data. It was found that the eects of model mis-specication on the likelihood estimation are serious in general. These results suggest the usefulness and the necessity of a model specication test for reliability assessment as well as risk management. On the other hand, generalized gamma distributions that include several popular lifetime distributions, namely, exponential, Weibull, gamma and log-normal distributions, was introduced by Stacy [12]. Due to its highly exibility, it is useful for analyzing lifetime data in reliability and survival analysis and discriminating among those models.

The generalized gamma distribution has been recently a great increase in practical application and interest. The problem on parameter estimation for the generalized gamma distributions has also been attempted by many researchers. Gomes et al. focused on the parameter estimation of the generalized gamma distribution. Noufaily and Jones [13] presented a comprehensive literature review on the maximum likelihood estimation of the parameters of the generalized gamma distribution and also proposed an iterative approach to solution of the likelihood score equations. However, methods for estimating parameters based on one-shot device testing data have not been studied. In this paper, we adopt two conventional methods of analysis for such data by nding estimates of the model parameters - Fisher scoring and least-squares methods. In addition to the model parameters, the reliability at a specic mission time and the mean lifetime under the normal operating condition are considered in this paper.

The reminder of this article is organized as follows. Section II describes the form of the one-shot device testing data under accelerated life-tests based on the generalized gamma distribution. In Section III, the Fisher scoring and the least-squares methods are developed for nding the estimates of the model parameters, as well as the reliability at a specic mission time and the mean lifetime under the normal operating condition. In Section IV, a simulation study is carried out for evaluating the performance of the proposed estimation methods for dierent levels of reliability and dierent sample sizes. Section V nally provides some concluding remarks.

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Generalized Gamma Model

In this section, the form of the one-shot device testing data under accelerated life-tests based on the generalized gamma distribution is described. Suppose that accelerated life-tests consist of I testing groups with J stress factors. For i =1,2,...,I,K_i one-shot devices are placed an elevated stress level $x_i=\{x_{i1}, x_{i2},..., x_{ij}\}$ under inspection time IT_i. The number of failures n_i is collected. The observed data can be summarized as in Table I.

Testing group	Inspection time	Number of tested devices	Number of failures	Covariates		
				Stress 1	...	Stress J
1	IT_1	K_1	n_1	x_{11}	…	x_{1J}
2	IT_2	K_2	n_2	x_{21}	…	x_{2J}
⋮	⋮	⋮	⋮	⋮	…	⋮
I	IT_I	K_I	n_I	x_{I1}	⋯	x_{IJ}

Table I: Data on one-shot device testing at various stress levels collected at dierent inspection times.

Let T denote a three-parameter generalized gamma random variable with the probability density function (pdf) of the form (see Stacy [12]) of

$$f(t,\alpha,\beta,\eta)=\frac{\eta t^{\beta-1}}{\Gamma(\beta/\eta)\alpha^{\beta}}\exp\left(-\left(\frac{t}{\alpha}\right)^{\eta}\right), t>0$$

where α > 0 is scale parameter, β > 0 and η > 0 are shape parameters. The corresponding cumulative distribution function (cdf) is given by

$$F(t,\alpha,\beta,\eta)=\int_0^t f(x,\alpha,\beta,\eta)dx=\frac{1}{\Gamma(\beta/\eta)}\int_0^{\omega} y^{\beta/\eta-1}\exp(-y)dy=\frac{\gamma(\beta/\eta,\omega)}{\Gamma(\beta/\eta)}$$

where $\omega=(t/\alpha)^{\eta}, \gamma(u,\upsilon)=\int_o^{\upsilon} y^{u-1}\exp(-y)dy$ is the lower incomplete gamma function, and Γ(u) gamma function. Moreover, the r-th moment of T and the reliability function at time t are, respectively,

$$E[T^{\gamma}]=\alpha\Gamma\left(\frac{\beta+r}{\eta}\right)/\Gamma\left(\frac{\beta}{\eta}\right)$$

and

$$R(t)=1-F(t)=\frac{\Gamma(\beta/\eta,\omega)}{\Gamma(\beta/\eta)},$$

where $w=(t/\alpha)^{\eta}, \Gamma(u,\upsilon)=\int_{\upsilon}^{\infty} y^{u-1}\exp(-y)dy$ is the upper incomplete gamma function.

It is noting that, the generalized gamma distribution becomes a two-parameter Weibull distribution with scale parameter and shape parameter η when β =η , it becomes a two-parameter gamma distribution with scale parameter α and shape parameter β when η = 1, and it becomes a one-parameter exponential distribution with scale parameter α when β = η = 1. Balakrishnan and Peng [15] presented another parametrization of the pdf of the generalized gamma distribution and its pdf is given by

$$g(y,q,\sigma,\lambda)=\begin{cases}|q|(q^{-2})^{q^{-2}}(\lambda y)^{q^{-2}(q/\sigma)}\exp(-q^{-2}(\lambda y)^{(q/\sigma)})/(y\sigma\Gamma(q^{-2})) & q\neq 0\\ (\sqrt{2\pi}\sigma y)^{-1}\exp(-(\log(\lambda y))^2/(2\sigma^2)) & q=0\end{cases}$$

with η=q/σ, $\beta=(q\sigma)^{-1}$ and $\alpha=\lambda^{-1} q^{(2\sigma/q)}$. Inversely we have $\sigma^2=(\beta\eta)^{-1}$, $q=\sqrt{\eta/\beta}$ and $\lambda=\alpha^{-1}(\eta/\beta)^{(1/\eta)}$. It can be seen that the generalized gamma distribution becomes a two-parameter log-normal distribution with location parameter μ=-log(λ) and scale parameter σ when q = 0.

Wang and Kececioglu [16] mentioned that many well-known stress-rate models, namely Arrhenius, inverse power law and Eyring, are all special cases of a log-linear model. For this reason, within each testing group, we further assume that all the three parameters are related to the stress factors in log-linear forms as

$$\alpha_i=\exp\left(\sum_{j=o}^{J}a_j x_{ij}\right),\ \beta_i=\exp\left(\sum_{j=o}^{J}b_j x_{ij}\right),\ \eta_i=\exp\left(\sum_{j=o}^{J}c_j x_{ij}\right),$$

where $x_{i0}\equiv 1$

In addition, the mean lifetime and the reliability at mission time t under the normal operating condition, $x_0 = \{x_j, j = 0, 1, 2,...,J\}$, are, respectively,

$$E[T\,|\,x_0]=\alpha_0\Gamma\left(\frac{\beta_o+1}{\eta_0}\right)/\Gamma\left(\frac{\beta_o}{\eta_0}\right)$$

and

$$R(t\,|\,x_0)=\frac{\Gamma(\beta_0/\eta_0,\omega_0)}{\Gamma(\beta_0/\eta_0)}$$

where $\omega_0=(t/\alpha_0)^{\eta_0}, \alpha_0=\exp\left(\sum_{j=o}^{J}a_j x_j\right), \beta_0=\exp\left(\sum_{j=o}^{J}b_j x_j\right)$, $\eta_0=\exp\left(\sum_{j=o}^{J}c_j x_j\right)$ and $x_0\equiv 1$

Point Estimation Methods

Balakrishnan and Ling [5] considered one-shot device testing data under Weibull distribution and investigated two popular estimation methods - the maximum likelihood estimation method and the least-squares estimation method - for finding the estimates of the model parameters. In this section, the two popular estimation methods for one-shot device testing data are described.

Maximum likelihood approach

The maximum likelihood estimation method is a general approach to nd the estimates of the model parameters by maximizing the log-likelihood function. The estimate of the model parameter is to be obtained as

$$\theta=\arg\max_{\theta} l(\theta,z)$$

In the present situation, the log-likelihood function is given by

$$l(\theta,z)=\sum_{i=1}^{I}n_i\log(F(IT_i,\alpha_i,\beta_i,\eta_i))+(K_i-n_i)\log(1-F(IT_i,\alpha_i,\beta_i,\eta_i))$$

where $z=\{K_i,n_i,IT_i,x_i,i=1,2,...,I\}$ is the observed data , and $\theta=\{a_j, b_j, c_j, j=0,1,2,...,J\}$ is the model parameters to be estimated.

Fisher scoring is a method to calculate the maximum likelihood estimates of the model parameters and solve the maximum likelihood equations numerically. The Fisher scoring method requires the score function V (θ), and the Fisher information matrix I_{obs} to solve the maximum likelihood equations. The score function is the rst-order derivative of the log-likelihood function with respect to the model parameters and is given by

$$v(\theta)=\begin{pmatrix}\dfrac{\partial(l(\theta))}{\partial a_j}\\ \dfrac{\partial(l(\theta))}{\partial b_j}\\ \dfrac{\partial(l(\theta))}{\partial c_j}\end{pmatrix}$$

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The Fisher information matrix is the covariance matrix of the score and a positive semidenite symmetric matrix. The Fisher information matrix is also known as the the second-order derivative of the log-likelihood function with respect to the model parameters. In our case, the Fisher information matrix is

$$I_{obs}(\theta) = -\begin{pmatrix} \frac{\partial^2(l(\theta))}{\partial a_p \partial a_q} & \frac{\partial^2(l(\theta))}{\partial a_p \partial b_q} & \frac{\partial^2(l(\theta))}{\partial a_p \partial c_q} \\ \frac{\partial^2(l(\theta))}{\partial a_p \partial b_q} & \frac{\partial^2(l(\theta))}{\partial b_p \partial b_q} & \frac{\partial^2(l(\theta))}{\partial b_p \partial c_q} \\ \frac{\partial^2(l(\theta))}{\partial a_p \partial c_q} & \frac{\partial^2(l(\theta))}{\partial b_p \partial c_q} & \frac{\partial^2(l(\theta))}{\partial c_p \partial c_q} \end{pmatrix}$$

The updated estimates of the model parameters θ is then determined as

$$\theta_{ML}^{(m+1)} = \theta_{ML}^{(m)} + I_{obs}^{-1}\left(\theta_{ML}^{(m)}\right) V \theta_{ML}^{(m)}$$

The expressions of the first and second order derivatives of the log-likelihood function with respect to the model parameters are presented in the Appendix. It is noting that a Taylor expansion of the score function is employed for nding the maximum likelihood estimates of the model parameters.

Least-Squares Approach

The least-squares estimation method is a general approach to approximate the solution of a system of equations by minimizing the sum of squares of errors between the observed and the expected values. In the present situation, the estimate of the model parameter is to be obtained as

$$\theta = \arg\min_{\theta} \sum_{i=1}^{I} \left(\frac{n_i}{K_i} - F(IT_i, \theta) \right)^2$$

Due to the non-linear form of $F_T(IT_i,\theta)$, there is no closed form solution to this non-linear least-squares problem. So, we make use of the Gauss-Newton method to approximate the solution iteratively. The updated estimates of the model parameters θ is then determined as

$$\theta_{LS}^{(m+1)} = \theta_{LS}^{(m)} + (J'^{(m)} J^{(m)})^{-1} J^{(m)} (p - F(IT, \theta_{LS}^{(m)}))$$

where $J^{(m)} = \left.\frac{\partial F(IT,\theta)}{\partial \theta}\right|_{\theta_{LS}^{(m)}}$

is the Jacobian matrix, and $p = \left(\frac{n_i}{K_i}\right)_{i=1,2,\dots,I}$

is a I × 1 vector.

Simulation Study

In this section, the performance of the proposed estimation methods for nding the estimates of the reliability at a specic mission time and the mean lifetime under the normal operating condition is assessed by means of a Monte Carlo simulation study, for dierent levels of reliability and dierent sample sizes, in terms of biases, root mean square errors (RMSE), and numbers of cases of convergence. Let ϕ denote a parameter of interest and $\hat{\phi}$ denote an point estimator for ϕ. The bias and RMSE are given by

$$Bias(\phi) = E[\hat{\phi}] - \phi, \text{ and } RMSE(\phi) = E[(\hat{\phi} - \phi)^2]$$

The lifetimes of devices were simulated from the generalized gamma distribution, under 12 dierent conditions with a single stress factor at 3 levels, taken to be {30,40,50}. Then, all devices under each condition were tested at 4 dierent inspection times. A balanced data with equal sample size for each group was considered. K_i was taken to be 50, 100 and 200, corresponding to small, medium, and large sample sizes, respectively. Since the generalized gamma distributions include two popular lifetime distributions gamma and Weibull distributions. The model parameters were set as (a_1, b_0, b_1, c_0, c_1)= (-0.06, -0.03, 0.04, 0, 0) for gamma distributions, and a_0 was chosen to be 4, 5, and 5.5, corresponding to devices with low, moderate and high reliability, respectively. The model parameters were set as (a_1, b_0, b_1, c_0, c_1)= (-0.05, -0.6, 0.03, -0.6, 0.03) for Weibull distributions, and a_0 was chosen to be 4.8, 5.3, and 5.7, corresponding to devices with low, moderate and high reliability, respectively. To prevent many zero-observations in test groups, the inspection times were not supposed to be the same for dierent levels of reliability. Specically, for both gamma and Weibull distributions, the inspection times were set as IT = (5, 10, 15, 20) for the case of low reliability, IT = (10, 20, 30, 40) for the case of moderate reliability, and IT = (15, 30, 45, 60) for the case of high reliability. The results obtained from the simulation study, based on 1,000 Monte Carlo simulations, are summarized in Tables II to V. The simulated values are calculated from the convergence cases. The numbers of cases of convergence are presented in Table VI.

The simulated results show that, for the maximum likelihood and the least-squares approaches, as the sample size increases, the estimates converge to the true values and the root mean square errors become small in both cases of the gamma andWeibull distributions. However, the maximum likelihood approach is more stable than the least-squares approach to yield accurate estimates of the mean lifetime and the reliability at mission time under the normal operating condition, in terms of biases and root mean square errors. Moreover, it is observed that the relative biases of the mean lifetime are generally greater than 0.2 in the cases of small samples. So, the sample size K≥100 is recommended when generalized gamma distribution is considered to t one-shot device testing test and estimate the mean lifetime under the normal operating condition. However, the bias and root mean square error on the reliability estimation are relatively small, compared with those on the mean lifetime, in the cases of small samples.

In addition, in Table VI, we can see that both maximum likelihood and least-squares approaches do not face serious convergence problem when sample sizes are suciently large. But the least-square approach yield enormous biases and root mean square errors in the cases of small sample sizes. In general, the maximum likelihood approach is comparably stable to nd the estimates of the model parameters.

Concluding Remarks

In this paper, one-shot device testing data that are subjected to either left or right censoring are considered. Generalized gamma distributions including many useful lifetime distributions, namely, exponential, gamma, Weibull, and log-normal distributions, are used to analyze one-shot device testing data. Two common estimation methods maximum likelihood and least squares approaches are compared to nd the estimates of the model parameters. The comprehensive simulation results show that both approaches are quite satisfactory for the estimation of the mean lifetime and the reliability at a specic mission time under the normal operating condition. The maximum likelihood approach outperforms the least squares approach to yield accurate estimates.

A0=4		E[T|x0]	R(10)	R(20)	R(30)	R(40)	R(50)
	Method	24.533	0.805	0.516	0.299	0.163	0.086
K=50	ML	2.991	0.006	0.019	0.018	0.024	0.030
	LS	4.717	0.008	0.023	0.019	0.028	0.035
K=100	ML	1.935	0.005	0.016	0.019	0.021	0.022
	LS	2.085	0.007	0.020	0.019	0.022	0.024
K=200	ML	0.968	0.002	0.006	0.008	0.009	0.011
	LS	62.376	0.000	0.009	0.014	0.013	0.014
A0=5		E[T|x0]	R(10)	R(20)	R(30)	R(40)	R(50)
	Method	66.686	0.964	0.879	0.774	0.664	0.559
K=50	ML	22.915	-0.005	0.003	0.012	0.016	0.014
	LS	27.171	-0.009	0.001	0.011	0.012	0.010
K=100	ML	8.687	-0.002	0.004	0.011	0.014	0.014
	LS	8.578	-0.004	0.004	0.011	0.014	0.011
K=200	ML	4.129	-0.001	0.001	0.004	0.006	0.006
	LS	4.879	-0.003	0.000	0.005	0.008	0.008
A0=5.5		E[T|x0]	R(10)	R(20)	R(30)	R(40)	R(50)
	Method	109.947	0.986	0.949	0.897	0.836	0.770
K=50	ML	31.847	-0.005	-0.001	0.005	0.010	0.013
	LS	469.481	-0.009	-0.007	0.001	0.008	0.009
K=100	ML	16.964	-0.003	-0.002	0.002	0.005	0.008
	LS	420.912	-0.005	-0.003	0.001	0.006	0.008
K=200	ML	9.764	-0.001	0.000	0.002	0.004	0.007
	LS	410.461	-0.003	-0.003	0.000	0.003	0.007

Table II: Biases of the estimates of the mean lifetime and the reliability at some mission times under normal operating condition x_0 = 25 for various choices of levels of reliability and sample sizes under the gamma distribution with $(a_1, b_0, b_1, c_0, c_1)$= (-0.06, -0.3, 0.04, 0, 0)

A0=4.8		E[T|x0]	R(10)	R(20)	R(30)	R(40)	R(50)
	Method	33.035	0.791	0.591	0.431	0.309	0.218
K=50	ML	19.208	0.010	0.033	0.042	0.038	0.042
	LS	39115.126	0.009	0.031	0.043	0.045	0.051
K=100	ML	7.712	0.005	0.022	0.029	0.027	0.027
	LS	181.845	0.004	0.021	0.029	0.030	0.032
K=200	ML	3.343	0.002	0.011	0.015	0.014	0.013
	LS	62.376	0.000	0.009	0.014	0.013	0.014
A0=5.3		E[T|x0]	R(10)	R(20)	R(30)	R(40)	R(50)
	Method	54.465	0.877	0.745	0.625	0.518	0.427
K=50	ML	15.771	-0.007	0.002	0.012	0.021	0.028
	LS	240071.785	-0.003	0.006	0.015	0.024	0.031
K=100	ML	9.519	-0.007	0.000	0.009	0.017	0.022
	LS	12.838	-0.005	0.000	0.008	0.014	0.020
K=200	ML	4.797	-0.004	0.001	0.007	0.011	0.014
	LS	6.203	-0.002	0.001	0.005	0.009	0.013
A0=5.7		E[T|x0]	R(10)	R(20)	R(30)	R(40)	R(50)
	Method	81.252	0.921	0.831	0.744	0.662	0.586
K=50	ML	26.904	-0.009	-0.005	0.003	0.011	0.019
	LS	556012.566	-0.006	0.000	0.007	0.016	0.023
K=100	ML	12.570	-0.006	-0.004	0.001	0.007	0.011
	LS	52.404	-0.004	-0.002	0.002	0.006	0.011
K=200	ML	7.720	-0.005	-0.003	0.000	0.005	0.008
	LS	9.134	-0.003	-0.002	0.001	0.004	0.007

Table III: Root mean square errors of the estimates of the mean lifetime and the reliability at some mission times under normal operating condition x_0 = 25 for various choices of levels of reliability and sample sizes under the Weibull distribution with $(a_1, b_0, b_1, c_0, c_1)$= (-0.05, -0.6, 0.03, -0.6, 0.03)

A0=4		E[T|x0]	R(10)	R(20)	R(30)	R(40)	R(50)
	Method	24.533	0.805	0.516	0.299	0.163	0.086
K=50	ML	8.641	0.054	0.079	0.107	0.109	0.099
	LS	36.741	0.062	0.089	0.122	0.123	0.104
K=100	ML	4.980	0.038	0.054	0.073	0.077	0.068
	LS	5.511	0.045	0.071	0.088	0.088	0.073
K=200	ML	3.238	0.025	0.036	0.051	0.055	0.049
	LS	3.512	0.029	0.038	0.055	0.060	0.053
A0=5		E[T|x0]	R(10)	R(20)	R(30)	R(40)	R(50)
	Method	66.686	0.964	0.879	0.774	0.664	0.559
K=50	ML	248.812	0.027	0.045	0.059	0.081	0.109
	LS	273.266	0.034	0.052	0.060	0.080	0.114
K=100	ML	37.023	0.019	0.031	0.041	0.057	0.078
	LS	27.074	0.024	0.038	0.046	0.060	0.083
K=200	ML	15.031	0.013	0.022	0.028	0.037	0.048
	LS	17.038	0.017	0.027	0.031	0.039	0.052
A0=5.5		E[T|x0]	R(10)	R(20)	R(30)	R(40)	R(50)
	Method	109.947	0.986	0.949	0.897	0.836	0.770
K=50	ML	127.579	0.019	0.032	0.041	0.048	0.057
	LS	903.165	0.026	0.040	0.050	0.056	0.060
K=100	ML	53.380	0.014	0.025	0.032	0.036	0.042
	LS	93.017	0.018	0.030	0.037	0.041	0.046
K=200	ML	30.776	0.008	0.016	0.022	0.027	0.033
	LS	31.970	0.011	0.020	0.025	0.029	0.033

Table IV: Root mean square errors of the estimates of the mean lifetime and the reliability at some mission times under normal operating condition x_0 = 25 for various choices of levels of reliability and sample sizes under the gamma distribution with (a_1, b_0, b_1, c_0, c_1)= (-0.06, -0.03, 0.04, 0, 0)

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	LS	62.376	0.000	0.009	0.014	0.013	0.014
		E[T|x0]	R(10)	R(20)	R(30)	R(40)	R(50)
A0=5.3	Method	54.465	0.877	0.745	0.625	0.518	0.427
K=50	ML	49.384	0.049	0.063	0.075	0.088	0.101
	LS	5697293.077	0.052	0.069	0.084	0.098	0.115
K=100	ML	28.257	0.036	0.044	0.051	0.060	0.070
	LS	49.718	0.036	0.044	0.052	0.061	0.072
K=200	ML	14.059	0.025	0.029	0.033	0.039	0.045
	LS	21.447	0.025	0.031	0.036	0.042	0.049
		E[T|x0]	R(10)	R(20)	R(30)	R(40)	R(50)
A0=5.7	Method	81.252	0.921	0.831	0.744	0.662	0.586
K=50	ML	76.276	0.041	0.054	0.062	0.070	0.078
	LS	13919729.233	0.043	0.058	0.068	0.078	0.087
K=100	ML	37.306	0.030	0.038	0.043	0.048	0.053
	LS	883.845	0.031	0.040	0.045	0.051	0.057
K=200	ML	21.629	0.021	0.026	0.028	0.030	0.033
	LS	32.599	0.022	0.027	0.031	0.034	0.038

Table V: Root mean square errors of the estimates of the mean lifetime and the reliability at some mission times under normal operating condition x0 = 25 for various choices of levels of reliability and sample sizes under the Weibull distribution with (a_1, b_0, b_1, c_0, c_1)= (-0.05, -0.6, 0.03, -0.6, 0.03)

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Gamma Level of reliability	K=50		K=100		K=200	
	ML	LS	ML	LS	ML	LS
High (a0=5.5)	938	967	934	952	934	965
Medium (a0=5)	907	955	935	959	940	958
Low (a0=5)	933	920	950	951	964	963
Weibull						
Level of reliability	K=50		K=100		K=200	
	ML	LS	ML	LS	ML	LS
High (a0=5.7)	970	938	994	968	998	988
Medium (a0=5.3)	968	937	991	968	997	988
Low (a0=4.8)	974	942	998	974	1000	1000

Table VI: Numbers of cases of convergence for various choices of levels of reliability and sample sizes under the gamma and Weibull distributions, based on 1000 simulations.

Appendix

Let $\theta=\{a_j, b_j, c_j, j=0, 1, 2,\ldots, j\}$ denote the model parameters to be estimated. Consider

$$F(IT_i,\theta)=\frac{\int_0^{\omega_i} u^{\frac{\beta_i}{\eta_i}-1}\exp(-u)du}{\Gamma\left(\frac{\beta_i}{\eta_i}\right)}$$

where $\omega_i=(IT_i/\alpha_i)^{\eta_i}$. The first order derivatives with respect to the model parameters are, respectively,

$$\frac{\partial F(IT_i,\theta)}{\partial a_j}=\frac{-\eta_i\omega_i^{\frac{\beta_i}{\eta_i}}\exp(-\omega_i)x_{ij}}{\Gamma\left(\frac{\beta_i}{\eta_i}\right)}$$

$$\frac{\partial F(IT_i,\theta)}{\partial b_j}=\frac{\beta_i x_{ij}}{\eta_i\Gamma\left(\frac{\beta_i}{\eta_i}\right)}\left(H_1\left(\frac{\beta_i}{\eta_i},\omega_i\right)-\gamma\left(\frac{\beta_i}{\eta_i},\omega_i\right)\psi\left(\frac{\beta_i}{\eta_i}\right)\right)$$

$$\frac{\partial F(IT_i,\theta)}{\partial c_j}=-\left(\left(\frac{\log(\omega_i)}{\eta_i}\right)\left(\frac{\partial F(IT_i,\theta)}{\partial a_j}\right)+\frac{\partial F(IT_i,\theta)}{\partial b_j}\right)$$

where

$$H_1(a,b)=\int_0^b \log(u)u^{a-1}\exp(-u)du=\log(b)\gamma(a,b)-\frac{b^a\,{}_2F_2(a,a;a+1,a+1;-b)}{a^2}$$

is the first order derivative of lower incomplete gamma function, and ${}_2F_2(a_1,a_2,b_1,b_2,z)$ is a Gaussian hypergeometric function that can be found by using Matlab, Maple and R.

Subsequently, the rst-order derivatives of the log-likelihood function with respect to the model parameters are

$$\frac{\partial l(\theta)}{\partial a_j}=\sum_{i=1}^{I}\left(\frac{n_i}{F(IT_i,\theta)}-\frac{K_i-n_i}{1-F(IT_i,\theta)}\right)\left(\frac{\partial F(IT_i,\theta)}{\partial a_j}\right),$$

$$\frac{\partial l(\theta)}{\partial b_j}=\sum_{i=1}^{I}\left(\frac{n_i}{F(IT_i,\theta)}-\frac{K_i-n_i}{1-F(IT_i,\theta)}\right)\left(\frac{\partial F(IT_i,\theta)}{\partial b_j}\right),$$

$$\frac{\partial l(\theta)}{\partial c_j}=\sum_{i=1}^{I}\left(\frac{n_i}{F(IT_i,\theta)}-\frac{K_i-n_i}{1-F(IT_i,\theta)}\right)\left(\frac{\partial F(IT_i,\theta)}{\partial c_j}\right)$$

Moreover, the second-order derivatives of the cdf with respect to the model parameters are, respectively,

$$\frac{\partial^2 F(IT_i,\theta)}{\partial a_p\partial a_q}=\frac{\eta_i\omega_i^{\frac{\beta_i}{\eta_i}}\exp(-\omega_i)(\beta_i-\eta_i\omega_i)x_{ip}x_{iq}}{\Gamma\left(\frac{\beta_i}{\eta_i}\right)}$$

$$\frac{\partial^2 F(IT_i,\theta)}{\partial a_p\partial b_q}=\frac{\beta_i\omega_i^{\frac{\beta_i}{\eta_i}}\exp(-\omega_i)\left(\psi\left(\frac{\beta_i}{\eta_i}\right)-\log(\omega_i)\right)x_{ip}x_{iq}}{\Gamma\left(\frac{\beta_i}{\eta_i}\right)}$$

$$\frac{\partial^2 F(IT_i,\theta)}{\partial a_p\partial c_q}=\frac{\omega_i^{\frac{\beta_i}{\eta_i}}\exp(-\omega_i)\left(-\eta_i+\eta_i\omega_i\log(\omega_i)-\beta_i\psi\left(\frac{\beta_i}{\eta_i}\right)\right)x_{ip}x_{iq}}{\Gamma\left(\frac{\beta_i}{\eta_i}\right)}$$

$$\frac{\partial^2 F(IT_i,\theta)}{\partial b_p\partial b_q}=\frac{\beta_i x_{ip}x_{iq}}{\eta_i\Gamma\left(\frac{\beta_i}{\eta_i}\right)}\left(H_1\left(\frac{\beta_i}{\eta_i},\omega_i\right)-\gamma\left(\frac{\beta_i}{\eta_i},\omega_i\right)\psi\left(\frac{\beta_i}{\eta_i}\right)\right)\left(1-\left(\frac{\beta_i}{\eta_i}\right)\psi\left(\frac{\beta_i}{\eta_i}\right)\right)$$
$$+\frac{\beta_i^2 x_{ip}x_{iq}}{\eta_i^2\Gamma\left(\frac{\beta_i}{\eta_i}\right)}\left(H_2\left(\frac{\beta_i}{\eta_i},\omega_i\right)-H_1\left(\frac{\beta_i}{\eta_i},\omega_i\right)\psi\left(\frac{\beta_i}{\eta_i}\right)-\gamma\left(\frac{\beta_i}{\eta_i},\omega_i\right)\psi'\left(\frac{\beta_i}{\eta_i}\right)\right)$$

$$\frac{\partial^2 F(IT_i,\theta)}{\partial b_p\partial c_q}=\left(\frac{\beta_i\log(\omega_i)\omega_i^{\frac{\beta_i}{\eta_i}}\exp(-\omega_i)x_{ip}x_{iq}}{\eta_i\Gamma\left(\frac{\beta_i}{\eta_i}\right)}\right)$$
$$\left(\log(\omega_i)-\psi\left(\frac{\beta_i}{\eta_i}\right)\right)-\frac{\partial^2 F(IT_i,\theta)}{\partial b_p\partial b_q}$$

$$\frac{\partial^2 F(IT_i,\theta)}{\partial c_p\partial c_q}=\left(\frac{\log(\omega_i)\omega_i^{\frac{\beta_i}{\eta_i}}\exp(-\omega_i)x_{ip}x_{iq}}{\eta_i\Gamma\left(\frac{\beta_i}{\eta_i}\right)}\right)$$
$$\left(\eta_i-\eta_i\log(\omega_i)\omega_i+\beta_i\psi\left(\frac{\beta_i}{\eta_i}\right)\right)-\frac{\partial^2 F(IT_i,\theta)}{\partial b_p\partial c_q}$$

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Where

$$H_2(a,b) = \int_0^b (\log(u))^2 u^{a-1} \exp(-u)$$

$$=(\log(b))^2 \gamma(a,b) - \frac{2\log(b) b^a {}_2F_2(a,a;a+1,a+1,-b)}{a^2}$$

$$+\frac{2b^a {}_3F_3(a,a,a;a+1,a+1,a+1,-b)}{a^3}$$

is the second-order derivative of lower incomplete gamma function.

Subsequently, the second-order derivative of the log-likelihood function with respect to θ_p, θ_q

$$\frac{\partial^2 l(\theta)}{\partial\theta_p \partial\theta_q} = \sum_{i=1}^{I} \left(\frac{\partial^2 F(IT_i,\theta)}{\partial\theta_p \partial\theta_q} \right) \left(\frac{n_i}{F(IT_i,\theta)} - \frac{K_i - n_i}{1-F(IT_i,\theta)} \right)$$

$$-\sum_{i=1}^{I} \left(\frac{\partial F(IT_i,\theta)}{\partial\theta_p} \right) \left(\frac{\partial F(IT_i,\theta)}{\partial\theta_q} \right) \left(\frac{n_i}{(F(IT_i,\theta))^2} + \frac{K_i - n_i}{(1-F(IT_i,\theta))^2} \right)$$

Conflict of interest

No authors have a conflict of interest or any financial tie to disclose.

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