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Controllability Problem for Linear Interval Systems

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Abstract

We consider controllability problem for linear dynamical systems with interval coefficients of corresponding matrices. We introduce concepts of universal and subuniversal controls, obtain methods of their construction and get their interesting properties.

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Introduction

The classical theory of controllability was introduced and developed in 1960-s by R. Kalman, L.S.Pontryagin, V.G.Boltyanski, R.V.Gamkrelidze, N.N.Krasovsky [1-4, 9]. Controllability problem is regarded as one of the basic for control theory and in general it means the possibility for given dynamical system to move from given position to another one for fixed time interval under appropriate control in corresponding vector space. We differ two types of controllability: point-to-point and total controllability. In this paper, our interest is focused on point-to-point controllability. Following [1, 3], point-to-point controllability is defined as

Definition

Let x^0, x^1 be two points in R^n and t_0, t_1 be two moments of time. Linear system

$$\dot{x} = A(t)x + B(t)u \qquad (1)$$

is said to be point-to-point controllable from position (x^0,t_0) to position (x^1,t_1) if there is a process (x(t),u(t)) such that

$$x(t_0)=x^0 \text{ and } x(t_1)=x^1$$

Traditionally $x(t) \in R^n$ is called a state (or phase) variable and $u(t) \in U \subset R^r$ is called a control variable or simply – control, A(t) and B(t) are $n \times n$ and $n \times r$ matrices accordingly. Here $t \in R$ is a time parameter.

Solution of point-to-point controllability problem in case $U = R^r$ has been obtained by Kalman [2, 4]. The following theorem gives the criterion of controllability.

Theorem

(Kalman). System (1) is controllable from position (x^0,t_0) to position (x^1,t_1) if and only if linear system (2) is consistent

$$W(t_0,t_1)z = x^1 - F(t_1,t_0)x^0 \qquad (2)$$

Here $z \in R^n$ is vector of unknowns, $W(t_0,t_1) = \int_{t_0}^{t_1} F(t_1,t)B(t)B^T(t)F^T(t_1,t)dt$ is an n×n matrix, F(t,τ) is a fundamental matrix of the system

$$\dot{x} = A(t)x,$$

satisfying initial condition

$x(\tau)=e^i$, where $e^i=(0...010...0)^T$ is i-th unit vector (i=1,2,...,n). We understand integration by elements. One of controls

$$u_c(t)=B^T(t)F^T(t_1,t)c$$

where c is a solution of (2) has an interesting property. It has a minimum norm

$$\| u \| = \left(\int_{t_0}^{t_1} u(t)^T u(t) dt \right)^{1/2}$$

in space $L_2^r(t_0,t_1)$ among all controls that solve point-to-point controllability problem.

Using of intervals is motivated by high level of uncertainty in real models, especially in economical models, under control. Limited possibility for observations and measurements, dynamic and non-stationary of processes make difficult the estimation of statistical characteristics of parameters, subjective probabilities and proper measures of fussy sets. In these conditions, well-known stochastic methods are not applicable and it is more preferable to utilize interval mathematical methods that assume knowing of only diapasons (intervals) of unknown parameters. Statistical functions of distribution of parameters inside the intervals are considered unknown. Interval approach is an effective tool for describing very wide circle of real problems. In mathematical modeling, interval methods are applied for analysis of uncertainty that arises when we use the data with errors, or we don't know the probability properties of objects under research, or errors of rounding in calculations with ultimate precision [13]. The result of applying interval models is the point evaluation of solution or the range of possible solutions. Mathematical instruments of interval computations ([12, 14, 15]) allow to formulate interval equations, interval problems of optimization and analyze interval functions.

This paper is devoted to solving point-to-point controllability problem for linear system (1) with interval coefficients of matrices A(t) and B(t) . First problem that we have to solve is the problem of getting solutions of interval linear system.

Auxiliary problem

Consider an interval linear system

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$$\sum_{j=1}^{n} a_{ij}x_j = b_i,\ i=1,2,\dots,m$$

where $\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, \underline{b}_i \le b_i \le \overline{b}_i$. Or in matrix form

$$Ax = b,$$
$$\underline{A} \le A \le \overline{A}, \underline{b} \le b \le \overline{b}. \qquad (3)$$

Problem

We need to determine vector x ∈ R^n satisfying system (3) for any matrices A and b, where $\underline{A} \le A \le \overline{A}, \underline{b} \le b \le \overline{b}$.

Introduce nonnegative vector ε as a discrepancy between vector Ax and vector b

|Ax-b|≤ε. Then

Definition

vector x ∈ R^n is called an ε -solution of system (3) if

b-ε ≤ Ax ≤ b+ε

for any matrices $A: \underline{A} \le A \le \overline{A}$ and $b: \underline{b} \le b \le \overline{b}$. Note that ε -solution always exists even when system (3) is not consistent.

According to [5,10] a necessary and sufficient condition for vector x ∈ R^n to be an ε - solution of system (3) is given by the following

Theorem 1

Vector x ∈ R^n is ε - solution of system (3) if and only if

$$\sum_{j=1}^{n} \max_{\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}} a_{ij}x_j \le \underline{b}_i + \varepsilon_i$$

$$\sum_{j=1}^{n} \max_{\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}} a_{ij}x_j \ge \overline{b}_i - \varepsilon_i,\quad i=1,2,\dots m.$$

Definition

ε -solution that has minimum norm is said to be a universal solution. Denote the *i-th* component of vector $\max_{\underline{A}\le A\le\overline{A}} Ax$ as $\sum_{j=1}^{n} \max_{\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}} a_{ij}x_j$ and the same for the *i-th* component of vector $\min_{\underline{A}\le A\le\overline{A}} Ax$: $\sum_{j=1}^{n} \min_{\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}} a_{ij}x_j$ and consider vector space R^m with the norm $\| z \| = \sum_{i=1}^{m} | z_i |$. Then a universal solution of system (3) can be obtained as a solution of the extreme problem

$$e^T\varepsilon \to \min$$
$$\max_{\underline{A}\le A\le\overline{A}} Ax \le \underline{b} + \varepsilon$$
$$\min_{\underline{A}\le A\le\overline{A}} Ax \ge \overline{b} - \varepsilon \qquad (4)$$
$$\varepsilon \ge 0$$

Here e^T=(1,1,...,1) is m-vector of one's.

A universal solution always exists as well as ε -solution. Problem (4) has at least one solution: x =0 and $\varepsilon \ge \max\{|\overline{b}|, |\underline{b}|\}$. Here and further we understand all matrices and vector operations by components.

Introduce another representation of intervals for components of matrix A: if we let

$$a_{0,ij} = \frac{1}{2}(\overline{a}_{ij} + \underline{a}_{ij}),\ a_{\Delta,ij} = \frac{1}{2}(\overline{a}_{ij} - \underline{a}_{ij}), \qquad (5)$$

then $a_{0,ij}-a_{\Delta,ij} \le a_{ij} \le a_{0,ij} + a_{\Delta,ij}$ is equivalent to $\overline{a}_{ij} \le a_{ij} \le \underline{a}_{ij}$.
In the light of notations (5) we have

$$\max_{\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}} a_{ij}x_j = \max_{|a_{ij}-a_{0,ij}| \le a_{\Delta,ij}} a_{ij}x_j$$

$$= \max_{|a_{ij}-a_{0,ij}| \le a_{\Delta,ij}} \left(a_{0,ij}x_j + \left(a_{ij}-a_{0,ij}\right)x_j\right) = a_{0,ij}x_j + a_{\Delta,ij}|x_j|$$

$$\min_{\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}} a_{ij}x_j = \min_{|a_{ij}-a_{0,ij}| \le a_{\Delta,ij}} a_{ij}x_j$$

$$\min_{|a_{ij}-a_{0,ij}| \le a_{\Delta,ij}} \left(a_{0,ij}x_j + \left(a_{ij}-a_{0,ij}\right)x_j\right) = a_{0,ij}x_j + a_{\Delta,ij}|x_j|$$

And analogue of theorem 1 is

Theorem 2

Vector x ∈ R^n is ε -solution of system (3) if and only if

$$\sum_{j=1}^{n} a_{0,ij}x_j + \sum_{j=1}^{n} a_{\Delta,ij}|x_j| \le \underline{b}_i + \varepsilon_i,$$

$$\sum_{j=1}^{n} a_{0,ij}x_j - \sum_{j=1}^{n} a_{\Delta,ij}|x_j| \le \overline{b}_i - \varepsilon_i \qquad (6)$$

$$i = 1,2,\dots,m$$

In more convenient matrix form condition (6) becomes

$$A_0x + A_\Delta|x| \le \underline{b} + \varepsilon,$$
$$A_0x - A_\Delta|x| \ge \overline{b} - \varepsilon. \qquad (7)$$

Here A_0 and A_Δ are m × n matrices of corresponding entries $a_{0,ij}$ and $a_{\Delta,ij}$. Due to notation (5) the extreme problem (4) obtains the explicit form

$$e^T\varepsilon \to \min$$
$$A_0x + A_\Delta|x| \le \underline{b} + \varepsilon$$
$$A_0x + A_\Delta|x| \ge \overline{b} + \varepsilon$$
$$\varepsilon \ge 0 \qquad (8)$$

Introduction of variable s= |x| allows us to reduce nonlinear problem (8) to linear problem (9)

$$e^T\varepsilon \to \min$$
$$A_0x + A_\Delta s \le \underline{b} + \varepsilon$$
$$A_0x - A_\Delta s \ge \overline{b} - \varepsilon$$
$$-s \le x \le s$$
$$\varepsilon \ge 0. \qquad (9)$$

Solution of linear problem (9) is a triple (x*, ε*, s*). x* is a universal solution and ε* is minimum discrepancy.

Sub-universal solution as an approximation of universal solution

Consider interval system (3) and assume that rank A_0 = m ≤ n. From (11) we obtain

ε ≥ ε(x)≡ A_Δ|x| + b_Δ

where $b_\Delta = \frac{1}{2}(\overline{b} - \underline{b})$

It is clear that pair (x, ε(x)) satisfies an equation

$$A_0x = b_0 \qquad (10)$$

where $b_0 = \frac{1}{2}(\overline{b} + \underline{b})$. And, conversely, any solution of (10) and corresponding vector ε≥ ε(x) are solutions of (7). Among all solutions of (10) our interest is related with normal solutions [6].

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Definition

Let us call a normal solution of (10) as a sub-universal solution. Normal solution of (10) has an explicit form

$$\hat{x} = A_0^+ b_0 \qquad (11)$$

where $A_0^+ = A_0^T (A_0 A_0^T)^{-1}$.

For sub-universal solution $\hat{x}$ we have $\hat{\varepsilon} = \varepsilon(\hat{x}) \equiv A_\Delta |\hat{x}| + b_\Delta$ It is straightforward task to show that $\hat{\varepsilon}$ has a minimum Euclidian norm.

Using a duality theory of linear problems [7] we obtain the relation between universal and sub-universal solutions:

$$0 \le e^T \hat{\varepsilon} - e^T \varepsilon^* \le e^T A_\Delta |\hat{x}|$$

In particular, if $A_\Delta = 0$, then universal and sub-universal solutions coincide.

Controllability problem for interval dynamics

Consider interval linear system

$$\dot{x} = A(t)x + B(t)u,$$

$$\underline{A}(t) \le A(t) \le \overline{A}(t), \underline{B}(t) \le B(t) \le \overline{B}(t) \qquad (12)$$

Here $x(t) \in R^n$ is a state variable, $u(t) \in U\ R^r$ = is control. Introduce matrices

$$A_0(t) = (\overline{A}(t) + \underline{A}(t))/2, B_0(t) = (\overline{B}(t) + \underline{B}(t))/2,$$

$$A_\Delta(t) = (\overline{A}(t) + \underline{A}(t))/2, B_\Delta(t) = (\overline{B}(t) + \underline{B}(t))/2$$

Then inequalities in (12) can be rewritten as

$$|A(t) - A_0(t)| \le A_\Delta(t), |B(t) - B_0(t)| \le B_\Delta(t) \qquad (13)$$

Let us call a deterministic system of the form

$$\dot{x} = A_0(t)x + B_0(t)u$$

centered in $A_0(t)$, $B_0(t)$ as a central system.

First, we give the outer evaluation of a beam of trajectories. Due to uncertainty of coefficients, the system (12) put into accordance to every control u(t) a family of trajectories corresponding to all feasible A(t),B(t) . We call a set of trajectories of the system (12) obtained for a fixed control u(t) and all feasible coefficients A(t),B(t) as a beam of trajectories corresponding to the given fixed control u(t) . A set $X(t_1)$ of right ends of phase trajectories $x(t_1)$ of the system (12) constructed for all feasible A(t),B(t) and fixed control u(t) is said to be a section of a beam of trajectories in the moment t_1.

Consider a particular case of the system (12):

$$\dot{x} = A(t)x, x(0) = x^0 \qquad (14)$$

assuming that $u(t) \equiv 0$, A(t) is some feasible matrix and x^0 is a fixed initial condition. For representation of a solution of (14), we use the matrix Cauchy problem

$$F_t(t,\tau) = A(t)F(t,\tau),$$
$$F(t,t) = I$$

(I is an identity matrix) and its fundamental matrix of solutions F(t, τ). Then

$$x(t) = F(t,0)x_0,\ t \ge 0.$$

We introduce special notations $F_0(t,\tau), \overline{F}(t,\tau), F_{|0|}(t,\tau)$ forfundamental matrices of the systems $\dot{x} = A_0(t)x, \dot{x} = (|A_0(t)| + A_\Delta(t))x, \dot{x} = |A_0(t)|x$ accordingly. Due to representation of F(t, τ) via matriciant [6] integral matrix series

$$F(t,\tau) = I + \int_\tau^t A(t_1)dt_1 + \int_\tau^t A(t_1)\int_\tau^{t_1} A(t_2)dt_2 dt_1 + \ldots$$

and evident inequality

$$|A(t)| \le |A_0(t)| + A_\Delta(t) \qquad (15)$$

we evaluate a solution of the system (14), considering a deviation of the beam of trajectories of the system (14) from the trajectory of the central system

$$\dot{x} = A_0(t)x, x(0) = x^0$$

Lemma

In notations for feasible matrices A(t) the following inequality holds:

$$|F(t,\tau) - F_0(t,\tau)| \le \overline{F}(t,\tau) - F_{|0|}(t,\tau) \equiv F_\Delta(t,\tau) \qquad (16)$$

One can easily prove this lemma applying the method of mathematical induction. Using inequality (16), we get the evaluation of a beam of trajectories of the system (14)

$$|x(t) - x_0(t)| \le F_\Delta(t,0)|x^0| \qquad (17)$$

Evaluation (17) possesses two important properties. Firstly, for $A_\Delta(t) \equiv 0$, from the definition of matriciants $\overline{F}(t,0)$, $F_{|0|}(t,0)$ we get $x \equiv x_0(t)$ that totally corresponds to the deterministic system (14) with a central matrix $A_0(t)$. Secondly, the evaluation (17) is decreasing, that is, $||F_\Delta(T,0)|| \to 0$ if $||A_\Delta(t)|| \to 0$ uniformly for $t \in [0,t_1]$.

Let us now evaluate a beam of trajectories of the general system (12) for some fixed control u(t) . For that we use the analogues to (15) evaluation of feasible coefficients

$$|B(t)| \le |B_0(t)| + B_\Delta(t)$$

We represent a solution x(t) of the system (12) by Cauchy formula [11] as

$$x(t) = F(t,0)x^0 + \int_0^t F(t,\tau)B(\tau)u(\tau)d\tau$$

From here, we obtain

$$|x(t) - x_0(t)| \le F_\Delta(t,0)|x^0|$$
$$+ \int_0^t F_\Delta(t,\tau)\left(|B_0(\tau)| + B_\Delta(\tau)\right) + F_0(t,\tau)|B_\Delta(\tau)||u(\tau)|d\tau \qquad (18)$$

Note that for deterministic system (12) with zero length of intervals, the evaluation (18) becomes the exact and it gives a trajectory of the central system.

We formulate the controllability problem as: the problem is to solve point-to-point controllability problem for system (12) subject to condition (13) for initial point $x^0 \in R^n$, terminal point $x^1 \in R^n$ and finite time interval $t \in [t_0,t_1]$

In other words, we have to determine a process (x(t),u(t)) such that $x(t_0)=x^0$ and $x(t_1)=x^1$ for any matrices A(t) and B(t) in (13).

Fix any matrices A(t) and B(t) in (13). Then according to Cauchy formula we get

$$x(t_1) = F(t_1,t_0)x(t_0) + \int_{t_0}^{t_1} F(t_1,t)B(t)u(t)dt, \quad u(t) \in R^r$$

If $x(t_0)=x^0$ and $x(t_1)=x^1$, then controllability problem is equivalent to validity of the following equality

$$x_1 = F(t_1,t_0)x_0 + \int_{t_0}^{t_1} F(t_1,t)B(t)u(t)dt \qquad (19)$$

for some pair (x(t),u(t)) .

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In turn, validity of (19) is equivalent to solvability of the system

$$\left(\int_{t_0}^{t_1} F(t_1,t)B(t)B^T(t)F^T(t_1,t)dt\right)\nu = x^1 - F(t_1 t)x^0 \qquad (20)$$

Solution of (20) is related with solution of (19) by u(t)= $B^T(t)F^T(t^1,t)$ ν. And the question of controllability of (12) is reduced to the question of solvability of the system (20). It is known [8, 9] that system (20) is consistent for any x^0 and x^1 if and only if the matrix

$$\left(\int_{t_0}^{t_1} F(t_1,t)B(t)B^T(t)F^T(t_1,t)dt\right)$$

is positively defined.

Let us make use of this criterion for solving point-to-point controllability problem when A(t) and B(t) are any matrices satisfying (13).

Construct a central system

$$\dot{x} = A_0(t)x + B_0(t)u$$

Then control $u_0(t)=B_0^T(t)F_0^T(t1,t)\nu$ moves central system from position (x^0,t_0) to position (x^1,t_1) where ν is a solution of

$$\left(\int_{t_0}^{t_1} F_0(t_1,t)B_0(t)B^T(t)F_0^T(t_1,t)dt\right)\nu = x^1 - F_0(t_1 t)x^0$$

and $F_0(t, \tau)$ is a fundamental matrix of corresponding system $\dot{x} = A_0(t)x$

We regard control $u_0(t)$ as a basis for solution of controllability problem. Then dynamical system (12), (13) is point-to-point controllable from position (x^0,t_0) to position (x^1,t_1) under control $u_0(t)$ if and only if the system

$$\left(\int_{t_0}^{t_1} F(t_1,t)B(t)B_0^T(t)F_0^T(t_1,t)dt\right)\nu = x^1 - F(t_1 t)x^0$$

has a solution.

Denote $D = \left(\int_{t_0}^{t_1} F(t_1,t)B(t)B_0^T(t)F_0^T(t_1,t)dt\right)$

Then due to evaluation (16) of a fundamental matrix F(t,τ), we get

$$D\nu = f, \underline{D} \le D \le \overline{D}, \underline{f} \le f \le \overline{f} \qquad (21)$$

Thus point-to-point controllability problem (12), (13) is reduced to solving of interval linear system (21).

Applying the concept of universal solution we form an extreme problem

$$\begin{aligned} & e^T\varepsilon \to \min \\ & D_0\nu + D_\Delta s - \varepsilon \le f_0 - f_\Delta \\ & -D_0\nu + D_\Delta s - \varepsilon \le -f_0 - f_\Delta \\ & -s \le \nu \le s \\ & \varepsilon \ge 0 \qquad (22) \end{aligned}$$

Here

$$D_0 = \left(\int_{t_0}^{t_1} F_0(t_1,t)B_0(t)B_0^T(t)F_0^T(t_1,t)dt\right), f_0 = x^1 - F_0(t_1,t)x^0$$

$$D_\Delta \le \int_{t_0}^{t_1} (F_\Delta(t_1,t)(|B_0(t)| + B_\Delta(t)) + |F_0(t_1,t)|\, B_0^T(t)F_0^T(t_1,t)\,|\,dt,$$

$$f_\Delta = F_\Delta(t_1,t_0)\,|\,x^0\,|, F_\Delta(t,\tau) = \overline{F}(t,\tau) - F_{|0|}(t,\tau)$$

Solution (ν*, ε*, s*) of the linear problem (22) allows us to form a universal control u*(t)= $B_0^T(t)F_0^T(t1,t)\nu^*$ that solves point-to-point controllability problem in the sense of ε solution.

Along with universal control we can construct a sub-universal control utilizing the explicit formula (11)

$$\hat{u}(t) = B_0^T(t)F_0^T(t_1,t)\hat{\nu} \qquad (23)$$

where $\hat{\nu} = D_0^{-1}f_0$ is a solution of the system

$$D_0\nu=f_0 .$$

Theorem 3

Sub-universal control (23) moves central system

$$\dot{x} = A_0(t)x + B_0(t)u$$

from the initial point x^0 to the terminal point x^1 for a finite interval $[t_0,t_1]$.

One can check the statement of this theorem by means of substitution of control $\hat{u}(t) = B_0^T(t)F_0^T(t_1,t)\hat{\nu}$ into the Cauchy formula.

Property 1

Sub-universal control has a minimum norm $\|u\| = \left(\int_{t_0}^{t_1} u(t)^T u(t)dt\right)^{1/2}$ in the space $L_2^r(t_0,t_1)$

This property follows from the classical theory of linear controlled systems [9].

Property 2

Sub-universal control is linear with respect to initial point x^0.

For proving of Property 2 it is sufficient to write a control in explicit form using formulas

$$D_0 = \left(\int_{t_0}^{t_1} F_0(t_1,t)B_0(t)B_0^T(t)F_0^T(t_1,t)dt\right),$$

$$\hat{\nu} = D_0^{-1}f_0 \text{ and } \hat{u}(t) = B_0^T(t)F_0^T(t_1,t)\hat{\nu}$$

Property 3

The outer evaluation of a section of a beam of trajectories on sub-universal control for $t=t_1$ contains the outer evaluation of a section of a beam of trajectories on universal control.

The proof follows from the feasibility of sub-universal solution in the linear programming problem (22).

Conflict of interest

No authors have a conflict of interest or any financial tie to disclose.

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