Amram, Int J Appl Exp Math 2017, 2: 119
http://dx.doi.org/10.15344/2456-8155/2017/119

International Journal of
Applied & Experimental Mathematics

Research Article Open Access

Topological Methods in Classification of Surfaces

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Abstract

In this paper we explain the steps for classification of algebraic surfaces. We compute the fundamental group of complements of branch curves in CP^2 and the fundamental group of the Galois covers of surfaces. We show the tight connection between these groups and Coxeter groups. Moreover, these groups are considered as invariants in the classification of surfaces.

Publication History:

Received: November 28, 2016
Accepted: July 15, 2017
Published: July 17, 2017

Keywords:

Algebraic surfaces, Geometric space, Classification of surfaces, Braid monodromy

The Moduli Space

In algebraic geometry, a moduli space is a geometric space whose points represent algebraic geometric objects (curves, surfaces) of some fixed type, or isomorphism classes of such objects.

If we can show that a collection of interesting surfaces induces the structure of a geometric space, then we can parameterize such surfaces by introducing coordinates on the resulting space.

In higher dimensions, moduli of algebraic surfaces are more difficult to construct and study.

In this paper, we give a comprehensive scientific background and list of methods to make the classification easier. Classification of surfaces is done by Catanese [7, 8], Kulikov [9, 10], Manetti [11], Moishezon-Teicher [12, 13].

Let $X \to CP^N$ be an embedded algebraic surface, $f: X \to CP^2$ be a generic projection of degree n. The branch curve of X in the plane CP^2 is S.

We compute the fundamental group $\pi_1(CP^2 - S)$, this group induces connected components in the moduli space: surfaces with the same group $\pi_1(CP^2 - S)$ are in the same connected component.

The ultimate goal of the classification is to compute the group $\pi_1(CP^2 - S)$, that is sometimes complicated. We try to find new invariants which distinguish connected components of the moduli space of surfaces (of general type). One of them is the fundamental group of the Galois cover of X.

The Set-up

In this section we describe the steps of the classification of algebraic surfaces, and follow the figure below:

(1) Degeneration of X to X_0.

(2) Projection of X_0 onto CP^2 to get S_0.

(3) Regeneration of S_0 to S.

(4) Braid monodromy technique of Moishezon-Teicher.

(5) Fundamental group of the complement of S.

(6) Fundamental group of the Galois cover of X.

$$\begin{tikzcd} X \subset CP^{N_1} \arrow[r, "\text{degeneration}"] \arrow[d, "f"'] & X_0 \subset CP^{N_0} \arrow[d] \\ X \subset CP^2 & X_0 \subset CP^2 \arrow[l, "\text{regeneration}"'] \end{tikzcd}$$

Step 1. Take a surface X embedded in CP^{N_1}, we degenerate X to a union of planes X_0. In order to understand how to do this, we give some examples of known degenerations.

Example 1. The surface $X = CP^1 \times CP^1$ (Moishezon-Teicher [13]).

The surface $CP^1 \times CP^1$ is defined by $z_1z_2 - 1z_0z_3 = 0 \to CP^3$. When $t = 0$ in $z_1z_2 - tz_0z_3 = 0$, we get $z_1z_2 = 0$, which is $CP^2 \cup CP^2$.

Therefore $X \xrightarrow{\text{degeneration}} X_0 = \bigcup^{\textit{number of planes}} CP^2$.

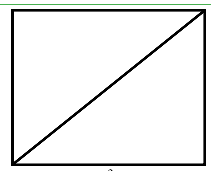

Figure 1: $X_0 = \overset{2}{\cup} CP^2$.

Example 2. The Veronese surface V_n (Amram-Lehman-Shwartz-Teicher [3]).

Here we give an example of the Veronese surface where n = 2:

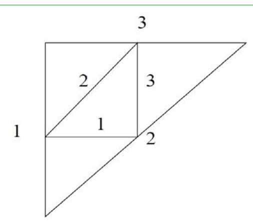

Figure 2: Degeneration X_0 of the Veronese V_2.

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Citation: Amram M (2017) Topological Methods in Classification of Surfaces. Int J Appl Exp Math 2: 119. doi: http://dx.doi.org/10.15344/2456-8155/2017/119

Example 3. The Hirzebruch surface $F_k(a, b)$ (Amram-Ogata [4]).

Degenerated Hirzebruch surfaces look like trapezoids. The parameter k shows the number of the triangles on the left and right sides of the trapezoids.

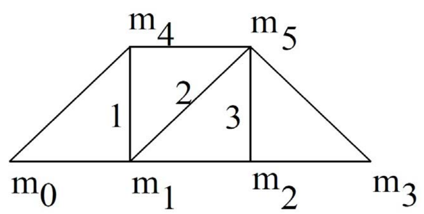

Figure 3: Degeneration of the Hirzebruch $F_2(1, 1)$.

Example 4. The surface $T \times T$ for T a complex torus (Amram-Teicher-Vishne [5]).

Each complex torus degenerates to a triangle, so the degenerated object has identifications along the exterior edges.

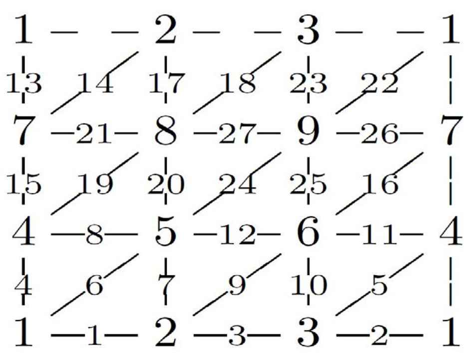

Figure 4: Degeneration of $T \times T$.

Example 5. The $K3$ surfaces (Amram-Ciliberto-Miranda-Teicher [1]).

The $K3$ surfaces degenerate to pillow objects, such that the top and bottom parts are identified along their exterior edges. Here we give an example of a (2, 2)-pillow.

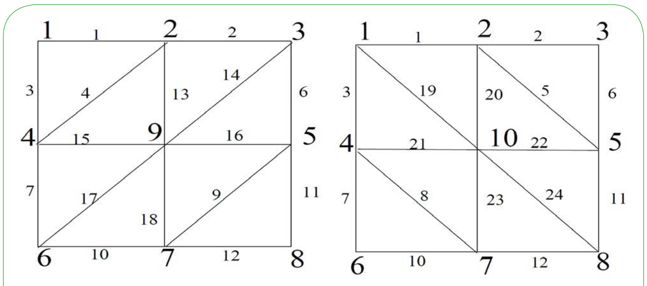

Figure 5: The (2, 2)-pillow degeneration.

Step 2. We project X_0 to the projective plane CP^2 to get the branch curve S_0.

In order to understand this, we take the following example:

Example 6. Let X_0 be the degenerated Hirzebruch surface $F_1(2, 2)$.

A generic projection $f_0 : X_0 \to CP^2$ is the degeneration of f. Under f_0, each of the 12 planes is mapped isomorphically to CP^2. The ramification locus R_0 of f_0 is composed of points in which f_0 is not isomorphism locally. Thus R_0 is the union of the 13 intersection lines. Let $S_0 = f_0(R_0)$ be the degenerated branch curve; it is a line arrangement, composed of the images of the 13 lines, see Figure 6.

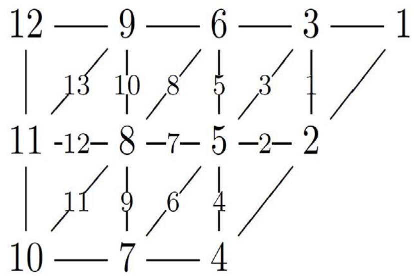

Figure 6: The degenerated Hirzebruch X_0.

Step 3. We regenerate the curve S_0 and recover the branch curve S. Regeneration Rules of Moishezon-Teicher [16, 17] on k-points are as follows:

2-points. The diagonal line 3 regenerates to a conic that is tangent to the line 1.

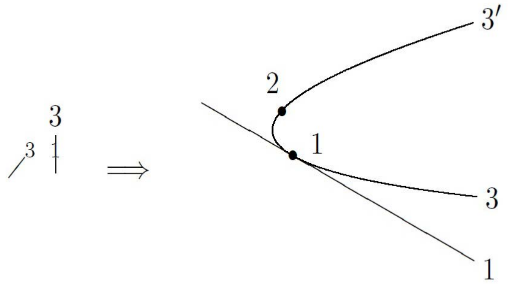

Figure 7: Regeneration of a 2-point.

3-points. We have two types of a 3-point. In each case, a diagonal line regenerates to a conic that is tangent to the other lines.

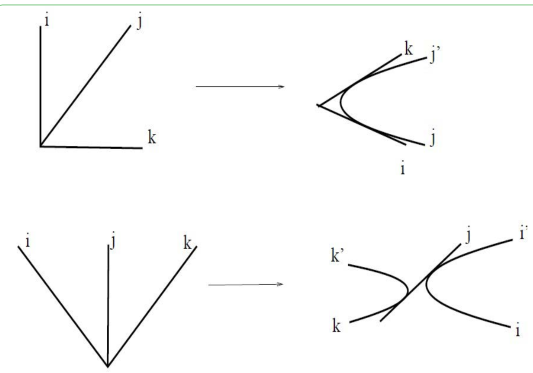

Figure 8: Regeneration of the two types of a 3-point.

6-points. The diagonal line 3 (resp. 6) regenerates to a conic that is tangent to the lines 2 and 5 (resp. 4 and 7). The other singularities are intersections.

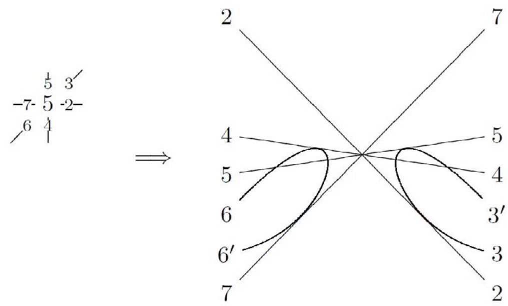

Figure 9: Regeneration of a 6-point.

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We are left with a 4-point in Figure 9. In Figure 10 we see how a 4-point regenerates to a hyperbola and two pairs of parallel tangent lines.

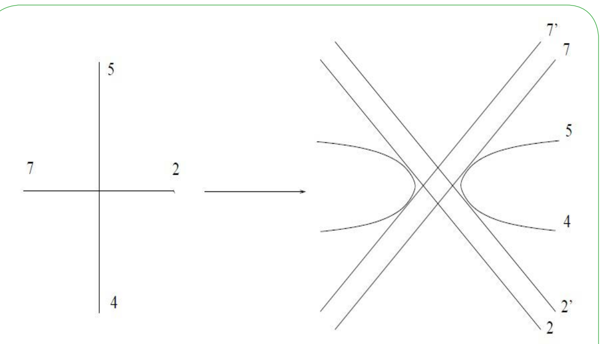


Figure 10: Regeneration of a 4-point.

Remark 1. In the next regeneration step, each tangency point regenerates to 3 cusps. Therefore S is a cuspidal curve.

Step 4. We compute the braid monodromy of Moishezon-Teicher [14, 15]. Here are the steps:

1. Find singularities in a curve S and take their x-coordinates x_i,
2. take "good" points M_i next to these x_i,
3. take loops around the x_i at M_i,
4. lift them and project to the fiber above M_i,
5. get a motion of the intersection points of S with the fiber over M_i.

Example 7. Consider the curve $y^2 = x^2$ and follow the above steps of the braid monodromy computation.

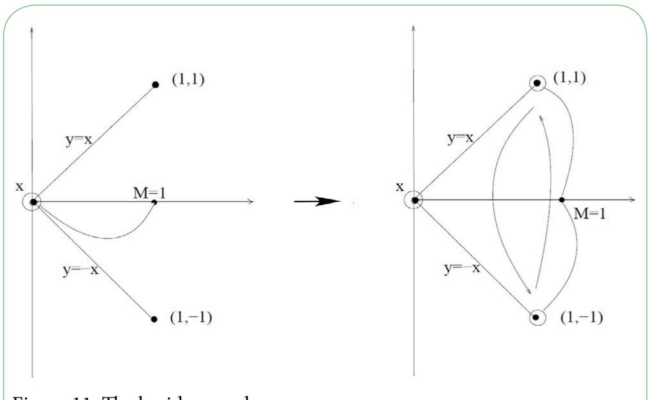


Figure 11: The braid monodromy.

Now we give a result of Moishezon-Teicher [14] concerning the monodromy computation.

Proposition 1. Take the curve S defined by $y^2 = x^m$. Then the braid monodromy is h^m, where h is a positive half-twist.

Proof. Take a tiny loop coming from the "good" point 1 to x_i, denoted by $x = e^{2\pi it}$, $t \in [0, 1]$. We lift this loop to the curve S and get two paths:

$$(e^{2\pi it}, e^{2\pi imt/2})$$

$$(e^{2\pi it}, -e^{2\pi imt/2}).$$

We project them onto the fiber above 1 and get two paths:

$$e^{\pi imt}$$
$$-e^{\pi imt}.$$

This gives the m-th power of the motion corresponding to $[-1, 1]$:

$$e^{\pi it}$$
$$-e^{\pi it}.$$

Step 5. We compute the group $\pi_1(\mathbb{CP}^2 - S) = \langle \Gamma_j \mid \{R\}\rangle$ (by means of generators and relations), it is the fundamental group of the complement of the branch curve S in $\mathbb{CP}^2$.

In order to produce elements of the fundamental group, we cut the braid at some point, then we produce two loops beginning and ending at the good point, circling the end points of the braid, see for example Figure 12.

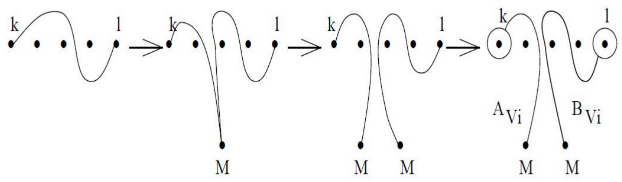


Figure 12: The elements of the fundamental group.

We give the Theorem of Zariski-van Kampen [19, 20] for cuspidal curves:

Theorem 1. *Let S be a cuspidal curve of degree n, then the fundamental group is generated by n generators and admits the following relations:*

(1) $\Gamma_i = \Gamma_{i+1}$ *for a branch point,*

(2) $[\Gamma_i, \Gamma_{i+1}] = \Gamma_i\Gamma_{i+1}\Gamma_i^{-1}\Gamma_{i+1}^{-1} = e$ *for a node,*

(3) $\langle \Gamma_i, \Gamma_{i+1}\rangle = \Gamma_i\Gamma_{i+1}\Gamma_i\Gamma_{i+1}^{-1}\Gamma_i^{-1}\Gamma_{i+1}^{-1} = e$ *for a cusp.*

In some cases the fundamental group $\pi_1(\mathbb{CP}^2 - S)$ might be very complicated, therefore we find new invariants. One of them is the fundamental group of the Galois cover of X. An example of a complicated $\pi_1(\mathbb{CP}^2 - S)$ is the one of $T \times T$ (Figure 4). In this case the group has 54 generators and around 2000 relations.

Step 6. We give a definition of the Galois cover of X ([13]) and show how to compute the fundamental group of a Galois cover of an algebraic surface.

Definition 1. We define the fibred product
$$\underbrace{X \times_f \ldots \times_f X}_{k} = \{(x_1, \ldots, x_k) \in X^k \mid f(x_1) = \ldots = f(x_k)\}$$
for $1 \le k \le n$, and the extended diagonal
$$\Delta = \{(x_1, \ldots, x_k) \in X^k \mid x_i = x_j \text{ for some } i \ne j\}.$$

The closure $X^k_{Gal} = \overline{X \times_f \ldots \times_f X - \Delta}$ is called the Galois cover with respect to the symmetric group.

Remark 2. There are many Galois covers with respect to the symmetric group S_k, $k < n$; but only for $k = n$ we identify the fundamental group of the Galois cover $\pi_1(X_{Gal})$ as the needed group. The Galois cover is a minimal smooth surface of general type.

Now we understand how to find the fundamental group of the Galois cover $\pi_1(X_{Gal})$. Consider the quotient group $\pi_1(\mathbb{CP}^2 - S)/\langle \Gamma_j^2 \rangle$ and take a canonical surjection from this quotient to the symmetric group S_n:

$1 \to \pi_1(X_{\mathrm{Gal}}) \to \pi_1(\mathrm{CP}^2\text{-}S)/\langle\Gamma_j^2\rangle \to S_n \to 1.$

The fundamental group $\pi_1(X_{Gal})$ is the kernel of this surjection, see Moishezon-Teicher [13].

Now we present various results of the computation of the fundamental group $\pi_1(X_{Gal})$.

(1) $X = \mathrm{CP}^1 \times \mathrm{CP}^1 \to \mathrm{CP}^3$ [13]:
$\pi_1(X_{Gal})$ is finite and commutative, $a \geq 3$, $b \geq 2$, and $\pi_1(X_{Gal}) = 0$ for a, b relatively prime.

(2) $X = \mathrm{CP}^1 \times T \to \mathrm{CP}^5$ for T a complex torus [2]:
$\pi_1(X_{Gal}) \cong \mathrm{Z}^{10}$; and in general, $\pi_1(X_{Gal}) \cong \mathrm{Z}^{4n-2}$.

(3) $X = T \times T \to \mathrm{CP}^8$ for T a complex torus [5]:
$\pi_1(X_{Gal})$ is nilpotent of nilpotency class 3 (there is a central series $G = H_1 \geq H_2 \geq \ldots \geq H_n = e$ such that each H_i is a normal subgroup of G and H_i/H_{i+1} is in the center of G/H_{i+1} and in our case $n = 3$).

(4) $X = F_1(2, 2)$ (Hirzebruch surface) [6]:
$\pi_1(X_{Gal}) \cong \mathrm{Z}_2^{10}$. In general, if $c = gcd(a, b)$ and $n = 2ab + kb^2$, then $\pi_1(X_{Gal}) \cong \mathrm{Z}_c^{n-2}$.

The Breakthrough to Coxeter and Artin Groups

We consist on the work of Rowen-Teicher-Vishne [18]. They define $C(T)$ to be a Coxeter group related to a graph T:

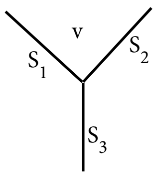
-generators s_i are the edges of T,
-relations

$s_i^2 = e \ \forall_i,$

$(s_i s_j)^2 = e$ if s_i, s_j are disjoint,

$(s_i s_j)^3 = e$ if s_i meets s_j in a vertex.

$C(T)$ has a natural map onto S_n, where n is the number of vertices of T. Then $C_Y(T)$ is a quotient of $C(T)$ and includes such cases: The "fork" relation is $(s_1 s_2 s_3 s_2)^2 = e, \forall v \in T$.



The main Theorem in [18] is the following:

$C_Y(T) \cong A_{t,n} \rtimes S_n$

where $A_{t,n}$ is a group which contains t copies of Z^{n-1}, n is the number of vertices of T, t is the number of cycles of T. The generators of the group are x_{ij}^r, $1 \leq r \leq t$, $1 \leq i, j \leq n$.

- $x_{ii}^r = e$, $1 \leq r \leq t$; $1 \leq i \leq n$
- $(x_{ij}^r)^{-1} = x_{ji}^r$, $1 \leq r \leq t$; $1 \leq i, j \leq n$
- $x_{ij}^r x_{jk}^r = x_{jk}^r x_{ij}^r = x_{ik}^r$, $1 \leq r \leq t$; $1 \leq i, j, k \leq n$ (not necessarily distinct)
- $x_{ij}^r x_{kl}^s = x_{kl}^s x_{ij}^r$, $1 \leq r$, $s \leq t$; $1 \leq i, j, k, l \leq n$ (distinct)

The connection to the classification of surfaces is explained using an example. Take the surface $T \times T$, where T is the complex torus. Projecting $T \times T$ *to* CP^2, we get the branch curve S. The group $\pi_1(\mathrm{CP}^2 - S)$ has 54 generators and admits around 2000 relations. We compute the quotient $\pi_1(\mathrm{CP}^2 - S)/\langle\Gamma_j^2\rangle$. The following exact sequence holds:

$1 \to \pi_1(X_{Gal}) \to \pi_1(\mathrm{CP}^2 - S)/\left\langle\Gamma_j^2\right\rangle \to S_{18} \to 1.$

And we get

$\pi_1(\mathrm{CP}^2 - S)/\left\langle\Gamma_j^2\right\rangle \cong \pi_1(X_{Gal}) \rtimes S_{18},$

where $\pi_1(X_{Gal})$ is nilpotent of class 3.

In general, there is a projection of the group $C_Y(T) \cong A_{t,n} \rtimes S_n$, on the group $\pi_1(\mathrm{CP}^2 - S)/\left\langle\Gamma_j^2\right\rangle \cong \pi_1(X_{\mathrm{Gal}}) \rtimes S_n$, so it is possible to calculate $\pi_1(X_{Gal})$ explicitly.

Competing Interests

The author declares that she has no competing interests.

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