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SSOR-Based Polynomial Preconditioner for Nonsymmetric Positive Definite Linear Systems

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Abstract

Krylov subspace methods, especially when combining with suitable preconditioners, are of great efficiency in solving large sparse linear systems. In this paper, based on symmetric successive over relaxation (SSOR) iterative method, an *m*-step polynomial preconditioner is designed and some theoretical results are presented. Numerical experiments demonstrate that the proposed preconditioner is effective in accelerating GMRES method for solving nonsymmetric positive definite linear systems.

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Introduction

Consider the system of linear equations

$$Ax = b, A \in R^{n\times n} \text{ and } x, b \in R^n. \quad (1.1)$$

Where is A a large, sparse, nonsingular matrix and b is a known vector. Numerical methods for solving linear system (1.1) can roughly be classified into sparse direct methods and iterative solvers. In general, direct solvers for large-scale problems are accurate, robust but memory consuming, see, for instance, [1,2]. Meanwhile, iterative methods involve only matrix-vector multiplications and hence can be attractive alternatives to direct solvers. Krylov subspace methods are a class of important iterative methods that are often applied for solving (1.1). Such methods include conjugate gradient method (CG) for symmetric positive definite coefficient matrices, see, for instance, [3]; symmetric LQ method (SYMMLQ) and minimum residual (MINRES) methods for symmetric indefinite coefficient matrices, see, for instance, [3,5-8]; generalized minimum residual (GMRES), biconjugate gradients stabilized (BiCGSTAB), quasi-minimum residual (QMR), BiCGCR2 that is anextension of biconjugate residual (BiCR) and biconjugate *A*-orthogonal residual (BiCOR) methods for nonsingular coefficient matrices, see, for instance, [4,9-13], and so forth. The preconditioned Krylov subspace method in which a preconditioner is combined with are widely used for solving large sparse linear systems because an efficient preconditioner can speed up the iteration greatly. A number of efforts have been devoted to the construction of suitable preconditioners when preconditioned Krylov subspace methods are used to solve the linear systems arising in different applications, see [3,4,13-15].

In this paper, we propose a polynomial preconditioner based on symmetric successive over relaxation (SSOR) method proposed in [18] when preconditioned Krylov subspace methods are used to solve the nonsymmetric positive definite linear systems (1.1). The outline of the paper is as follows. In section 2,we review alternating iterative methods and describe polynomial preconditioners succinctly; the construction of a new polynomial preconditioner is presented in section 3; In section 4, some numerical experiments are carried out to demonstrate the effectiveness of the proposed method; concluding remarks are given in the last section.

Preliminaries

In this section, alternating iterative methods and polynomial preconditioners are introduced. They are foundations of the proposed SSOR-based *m*- step polynomial preconditioners.

Alternating Iterative Methods

Alternating iterative methods have been widely used for solving the linear system (1.1), see, for instance, [4,15-17]. Let $A=M_i-N_i(i=1,2)$ be two splittings of the coefficient matrix, where M_i $(i=1,2)$are nonsingular. For an initial vector $x^{(0)}$, an alternating iterative method is implemented as follows:

$$\begin{cases} M_1 x^{(k+\frac{1}{2})} = N_1 x^{(k)} + b, \\ M_2 x^{(k+1)} = N_2 x^{(k+\frac{1}{2})} + b. \end{cases} \qquad k = 0,1,2,... \quad (2.1)$$

By bringing $x^{(k+\frac{1}{2})}$ into the second step of (2.1), we have

$$x^{(k+1)} = M_2^{-1}N_2M_1^{-1}N_1x^{(k)} + M_2^{-1}(M_1 + N_2)M_1^{-1}b \quad (2.2)$$

If M_1+N_2 is nonsingular, let $G = M_2^{-1}N_2M_1^{-1}N_1$, $M=M_1(M_1+M_2)^{-1}M_2$and $N=MG$. Then (2.2) is an equivalent stationary iterative scheme to (2.1) for splitting $A=M-N$. (2.1) converges if and only if $\rho(G)<1$, where $\rho(.)$ denotes the spectral radius of the corresponding matrix.

The following Lemma, refer to [18,19], shows that the alternating iteration is convergent if the splittings satisfy some special conditions.

Lemma 2.1 Let $A \in R^{n\times n}$ be a nonsymmetric positive definite matrix, $A=M_1-N_1=M_2-N_2$ are both p-regular splittings with N_1 and N_2

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being both symmetric. Denote $G= M_2^{-1}N_2M_1^{-1}N_1$, then $\rho(G)<1$.

$A=M-N$ is called a p-regular splitting if and only if M is nonsingular and $M^T + N$ is positive definite. The proof details of Lemma 2.1 can be found in [18,19]. Lemma 2.1 shows that when A is nonsymmetric positive definite, the alternating iterative method is convergent for any starting vector $x^{(0)}$ if both splittings in alternating iterative procedure are p-regular with N_1 and N_2 being both symmetric.

Decompose the coefficient matrix A in (1.1) as

$$A = D - L - U \tag{2.3}$$

where D, $-L$, $-U$ are the diagonal, the strictly lower triangular and strictly upper triangular parts of A. Specifically, we consider the following SSOR iteration proposed in [18] for nonsymmetric positive definite linear systems (1.1).

$$\begin{cases} (\frac{1}{\omega}D - L + U^T)x^{(k+\frac{1}{2})} = [(\frac{1}{\omega}-1)D + U + U^T]x^{(k)} + b, \\ (\frac{1}{\omega}D - U + L^T)x^{(k+1)} = [(\frac{1}{\omega}-1)D + L + L^T]x^{(k+\frac{1}{2})} + b, \end{cases} \tag{2.4}$$

Where ω is a positive constant. Comparing with formula (2.1), the splittings $A=M_i-N_i(i=1,2)$ in (2.4) are with

$$M_1 = \frac{1}{\omega}D - L + U^T,\ N_1 = (\frac{1}{\omega}-1)D + U + U^T \tag{2.5}$$

and

$$M_2 = \frac{1}{\omega}D - U + L^T,\ N_1 = (\frac{1}{\omega}-1)D + L + L^T \tag{2.6}$$

in (2.4). Suppose that $M_1 + N_2 = \frac{2}{\omega}D - A$ is nonsingular and define

$$\begin{aligned} G(\omega) = &[D - \omega(U - L^T)]^{-1}[(1-\omega)D + \omega(L + L^T)] \\ &\bullet[D - \omega(L - U^T)]^{-1}[(1-\omega)D + \omega(U + U^T)] \end{aligned} \tag{2.7}$$

$$M(\omega) = (\frac{1}{\omega}D - L + U^T)(\frac{2}{\omega}D - A^T)(\frac{1}{\omega}D - U + L^T) \tag{2.8}$$

Then SSOR iterative method (2.4) is equivalent to the stationary iterative procedure $x^{(k+1)} = G(\omega)x^{(k)} + M(\omega)^{-1}b$ with the splitting $A=M(\omega)+N(\omega)$, where $N(\omega) = M(\omega)\,G(\omega)$.

The choice of relaxation factor ω in (2.4) is not necessarily easy, and depends upon the properties of the coefficient matrix. The following theorem shows that the SSOR iteration (2.4) is convergent when $\omega \in (0,1)$for special coefficient matrices.

Theorem 2.1: Let $A \in R^{n\times n}$ be a nonsymmetric positive definite matrix with $H =(A+A^T)$ being its symmetric part. Split $A=D-L-U$ where D,-L-U are the diagonal, the strictly lower triangular and strictly upper triangular parts of A. Let $\eta =\lambda_{\min}(B)$ and $\mu= \lambda_{\min}(B)$ be the minimum eigenvalues of $B=H+2(U+U^T)$ and $C = H+2(L+L^T)$, respectively. If $\eta\geq 0$ and $\mu\geq 0$, then when $\omega \in (0,1)$, the SSOR iteration (2.4) is convergent.

Proof : Consider the splitting $A=M_1-N_1$, where M_1 and N_1 are shown in (2.5), M_1 is nonsingular and N_1 is symmetric. Let $H(M_1)=(M_1+M_1^T)/2$, then

$$\begin{aligned} H(M_1) + N_1 = H + 2N_1 &= \frac{2-2\omega}{\omega}D + H + 2(U + U^T) \\ &= \frac{2-2\omega}{\omega}D + B \end{aligned} \tag{2.9}$$

Since $\eta \geq 0$ and $\omega \in (0,1)$, $H(M_1)+N_1$ is symmetric positive definite. Therefore, $M_1^T + N_1$ is positive definite, i.e., $A=M_1-N_1$ is a p-regular splitting of A. Similarly, $A=M_2-N_2$ where M_2 and N_2 are shown in (2.6) is also a p-regular splitting of A. Combining Lemma 2.1, SSOR iteration (2.4) is convergent.

Polynomial Preconditioners

Let $A=M-N$ be a splitting of A and $\rho(M^{-1}N)<1$. Define $G=M^{-1}N$, in view of the fact that $M^{-1}A=I-G$ and $\rho(G)<1$, we have.

$$A^{-1} = (I - G)^{-1}M^{-1} = (\sum_{k=0}^{\infty} G^k)M^{-1}$$

Denote

$$P_m = M(\sum_{k=0}^{m-1} G^k)^{-1}, \quad m \geq 1 \tag{2.10}$$

P_m can be applied as a preconditioner to A and we call it an m-step polynomial preconditioner. P_m can also be induced from the two-stage iteration methods of trivial outer splittings; see [20-23]. Since $\rho(G)<1$, $I-G^m$ and $\sum_{k=0}^{m-1} G^k$ are nonsingular. Therefore, P_m is well defined.

In the preconditioned Krylov subspace methods, the main computation is to solve a linear system with the coefficient matrix being the preconditioner. For example, if we combine preconditioner to P_m GMRES method, we need to solve a linear system $P_m z = r$. It follows directly from (2.10) that

$$z = P_m^{-1}r = (I + G + \ldots + G^{m-1})M^{-1}r.$$

The algorithm for calculating z is depicted in Algorithm 1.

Algorithm 1 Solving $P_m z = r$.

1: Input $Z^{(0)}=0$

2: For k=1,...,m do

$$Mz^{(k)} = Nz^{(k-1)} + r$$

End for

3: Output $z=z^{(m)}$.

The Polynomial Preconditioners Based on SSOR

By utilizing SSOR method introduced in Section 2.1, we propose the following m- step polynomial preconditioner

$$P_m(\omega) = M(\omega)\cdot(I + G(\omega) + \ldots + [G(\omega)]^{m-1})^{-1}, \quad m \geq 1. \tag{3.1}$$

Where $G(\omega)$ and $M(\omega)$ are defined in (2.7) and (2.8), respectively. We term $P_m(\omega)$ as SSOR(m). SSOR(m) preconditioned GMRES method is particularly symbolized as SSOR(m)-GMRES. The SSOR(m)-preconditioned matrix can be expressed as

$$P_m(\omega)^{-1}A = I - [G(\omega)]^m.$$

From Algorithm 1, we know when m increases, more linear systems with same coefficient matrix $M(\omega)$ should be solved. However, the following theorem shows that the eigenvalues of the SSOR(m)-preconditioned matrix are more clustered if m is bigger, and thus the convergence may be faster. In the subsequent of the paper, $\lambda(.)$ denotes the set of eigenvalues of the corresponding matrix.

Theorem 3.1 Let $G(\omega)$ be defined in (2.7) and $\rho(G(\omega))<1$, then the eigenvalues of the SSOR(m)-preconditioned matrix $P_m(\omega)^{-1}A$ are

located in a disk centered at (1,0) with radius $\rho(G(\omega))^m$.

Proof: Suppose λ is an eigenvalue of $G(\omega)^m$ and is an eigenvector of λ, that is,

$$G(\omega)^m x = \lambda x$$

λ locates in $B((0,0))$, $\rho(G(\omega)^m)$ where $B((a,b),r)$ denotes a disk with point (a,b) being its center and r being its radius. Since $[P_m(\omega)^{-1}A]x = [I - G(\omega)^m]x = (1-\lambda)x$, then $\forall \lambda' \in \lambda(P_m(\omega)^{-1}A)$, we have $\lambda' \in B((1,0), \rho(G(\omega)^m)) = B((1\ 0), \rho(G(\omega))^m)$.

Since $\rho(G)<1$, Theorem 3.1 shows that the larger m is, the smaller spectral radius of the preconditioned matrix is, and thus we can expect a faster convergence. However, the computational cost becomes higher when m increases. How to select a suitable value of m is difficult and problem dependent.

Numerical Experiments

In this section, we demonstrate the performance of the proposed SSOR-based polynomial preconditioner when used in GMRES method to solve some examples. All codes are run in MATLAB R2013a (version 8.1.0.604) on a PC configured with an Intel(R) 3.4G CPU and 8GB RAM. The initial guess is taken to be zero, the right-hand side is a column vector which entries are sums of corresponding rows of the coefficient matrix A. The iteration is terminated once the residual satisfies

$$r = \frac{\left\|b - Ax^{(k)}\right\|_2}{\left\|b - Ax^{(0)}\right\|_2} \le 10^{-6}$$

Example 4.1: *For a fixed n, let* $I \in R^{nxn}$ *be the identity matrix of order* n and $E \in R^{nxn}$ of the form

$$E = 100 \cdot I + 90 \cdot E_1 / \sqrt{n}$$

Where E_1 is an n-by-n-random matrix generated by the function randn() in matlab. $D \in R^{nxn}$ is a diagonal matrix whose diagonal elements are

$$d_k = 100 - 100 \cdot \sin(\frac{k\pi}{n-1}), \quad k = 0, \ldots, n-1.$$

We consider linear system (1.1) with coefficient matrix $A=E+D$.

Example 4.1 is not an increasing denser discretization of a continuous model. The computing results of Example 4.1 are shown in Table 1. Where "IT" denotes the iteration steps, "TIME" denotes the running time in milliseconds which is shown in parentheses after "IT". We just show the results for m=1,2,...,5 in SSOR(m)-GMRES method. Table 1 tells us that if m is larger, the iteration number is smaller. But when m≥5 , the computing time will not decrease, which is in accordance with the analysis in Theorem 3.1. $m=4$ could be the best choice for this example. Table 2 shows the optimal relaxation parameter ω_{opt} used in SSOR(m)-GMRES method. In our experiments, we find the optimal ω is always located in range (0,1) and ω_{opt} is determined by trial and error. That is, we searched over a range from 0.001 to 0.999 with step 0.001, and the optimal ω is determined which makes the iterative steps smallest.

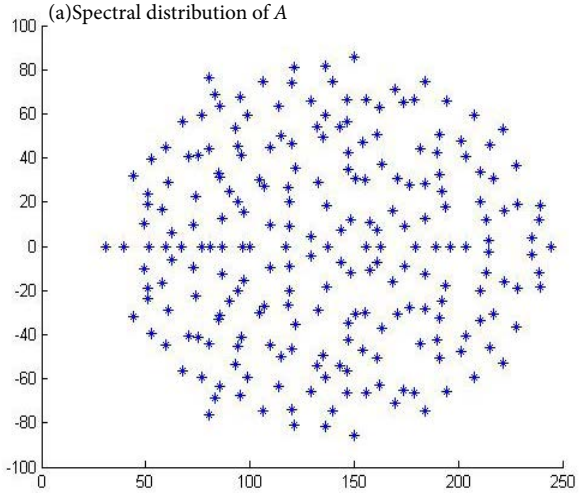
The spectral distributions of the coefficient matrix A and the preconditioned matrix for Example 4.1 when n=225 are shown in Figure 1. They are totally consistent with the conclusions of Theorem 3.1.

IT(TIME) / Method \ n	100	225	400	625	900
GMRES	29(3.86)	35(4.22)	36(1.28)	37(1.67)	38(2.40)
SSOR(1)-GMRES	14(0.35)	17(0.63)	17(1.15)	17(2.97)	17(5.58)
SSOR(2)-GMRES	8(0.23)	9(0.45)	10(1.05)	9(2.74)	9(5.40)
SSOR(3)-GMRES	6(0.21)	7(0.46)	7(1.02)	7(3.01)	7(6.14)
SSOR(4)-GMRES	4(0.18)	5(0.43)	5(0.97)	5(2.91)	5(6.01)
SSOR(5)-GMRES	4(0.20)	4(0.43)	4(0.97)	5(3.47)	4(6.19)

Table 1: Computing Results of Example 4.1.

ω_{opt} / Method \ n	100	225	400	625	900
SSOR(1)-GMRES	0.583	0.798	0.802	0.609	0.686
SSOR(2)-GMRES	0.896	0.812	0.651	0.724	0.785
SSOR(3)-GMRES	0.873	0.875	0.833	0.860	0.851
SSOR(4)-GMRES	0.823	0.833	0.827	0.829	0.789
SSOR(5)-GMRES	0.837	0.856	0.861	0.886	0.843

Table 2: ω_{opt} in SSOR(m)-GMRES for Example 4.1



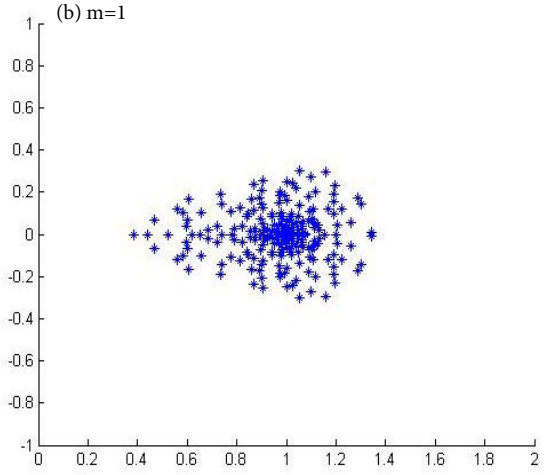


Figure 1: Continue...

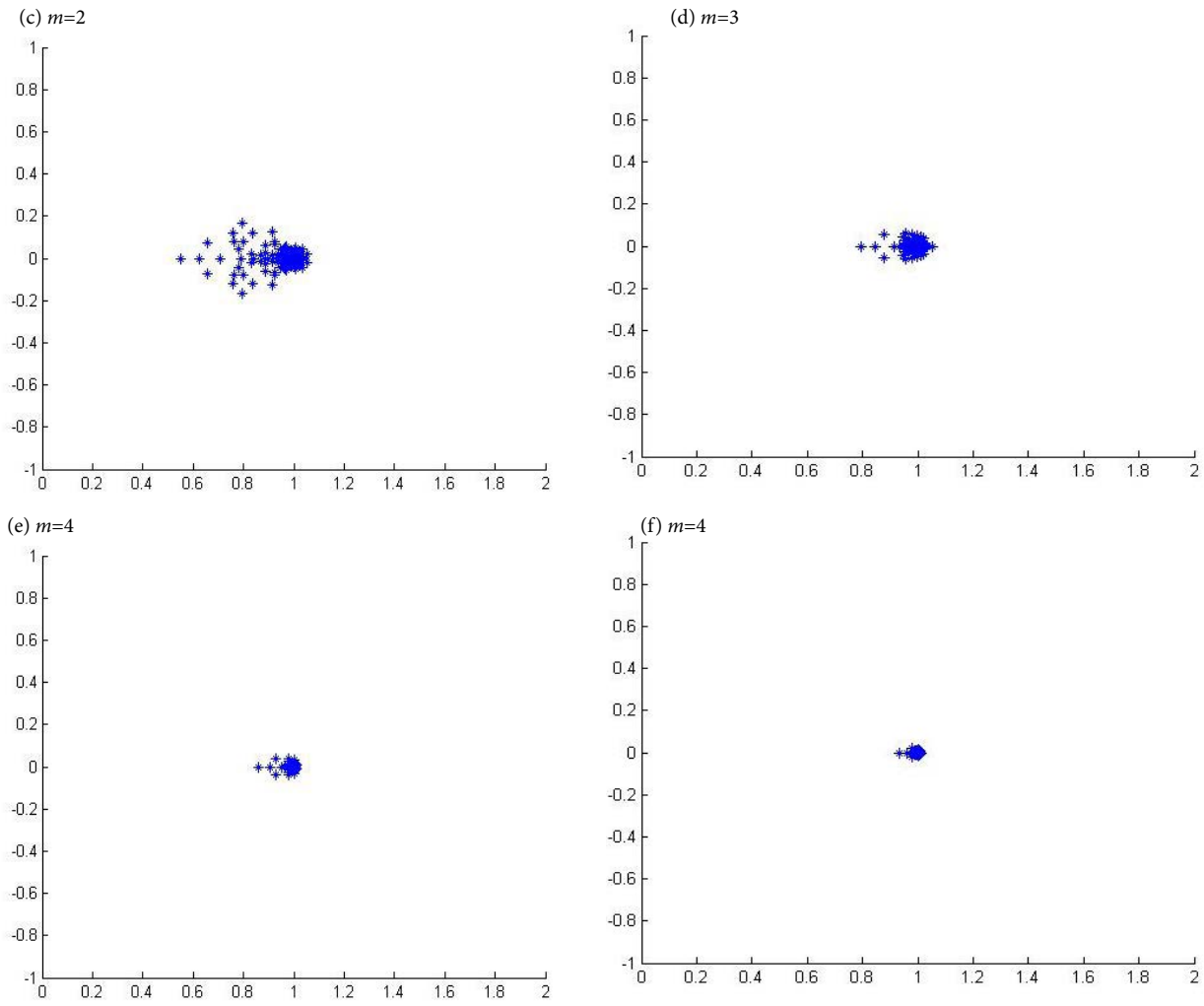


Figure 1: Spectral distribution of $p_m(\omega)^{-1}A$ for Example 4.1

Example 4.2 (see [24]) Consider the augmented linear systems (1.1) with coefficient matrix

$$A=\begin{pmatrix} B & E \\ -E^T & \mu I \end{pmatrix}$$

Where

$$B=\begin{pmatrix} I\otimes T_H + T_H \otimes I & 0 \\ 0 & I\otimes T_H + T_H \otimes I \end{pmatrix}$$

and

$$E=\begin{pmatrix} I\otimes F \\ F\otimes I \end{pmatrix}$$

$\otimes$ represents the Kronecker product, T_H and F are tridiagonal matrices as follows:

$$T_H = \text{tridiag}(-1,2,-1),\ F=\delta h\cdot \text{tridiag}(-1,1,0)$$

Discretization step size $h=\frac{1}{n+1}$.

Same as in [24], we perform experiments when μ=0.5 and δ =10 in this example. Computing results are listed in Table 3. The computing effects are similar to those of Example 4.1. If m is larger, the iteration number is smaller. But when m≥ 5 , the computing time will not decrease. m=4 could be the best choice for this example. Table 4 displays the experimental optimal parameter ω_{opt} in SSOR(m)-GMRES.

IT(TIME) n / Method	8	16	24	32	40
GMRES	31(0.66)	43(1.67)	63(5.03)	79(23.87)	98(44.84)
SSOR(1)-GMRES	12(0.26)	17(0.74)	21(1.99)	26(6.54)	32(13.84)
SSOR(2)-GMRES	8(0.19)	12(0.63)	16(1.87)	19(5.71)	23(12.25)
SSOR(3)-GMRES	7(0.18)	10(0.61)	13(1.81)	15(5.35)	19(11.83)
SSOR(4)-GMRES	6(0.17)	9(0.60)	12(1.80)	14(5.35)	16(11.68)
SSOR(5)-GMRES	5(0.16)	8(0.60)	11(1.85)	12(5.27)	15(11.56)

Table 3: Computing Results of Example 4.2.

ω_{opt} n / Method	8	16	24	32	40
SSOR(1)-GMRES	0.992	0.887	0.990	0.983	0.990
SSOR(2)-GMRES	0.980	0.965	0.976	0.928	0.979
SSOR(3)-GMRES	0.958	0.946	0.954	0.996	0.954
SSOR(4)-GMRES	0.964	0.899	0.966	0.922	0.999
SSOR(5)-GMRES	0.937	0.919	0.968	0.987	0.986

Table 4: ω_{opt} in SSOR(m)-GMRES for Example 4.2.

Conclusions

In this paper, we proposed an m-step polynomial preconditioner based on SSOR when preconditioned Krylov subspace methods such as GMRES method is used for solving the nonsymmetric positive definite linear systems. Theoretical analysis and numerical experiments demonstrate that the proposed preconditioner is efficient to be used in GMRES for solving nonsymmetric positive definite linear systems.

Competing Interests

The authors declare that they have no competing interests.

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