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Market Equilibrium as a Constrained Optimal Solution

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Abstract

Following Negishi (1960), we characterize the competitive equilibrium as the saddle point of a (constrained) Lagrangian and prove its existence for the case of finite dimensional spaces. We also outline how to extend this method to infinite dimensional spaces as a variational problem employing Dirac's delta function.

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Introduction

The mathematical theory of economic markets was established in the mid-1950s [1,20,22]. This describes an equilibrium state ("market equilibrium" or "competitive equilibrium"; see Definition 1) resulting from trades for finitely many commodities between finitely many traders (consumers) where consumers each achieve their best satisfaction within their own economic condition (budget constraint). Hence, no one would deviate from the current position, but only as long as economic or environmental changes do not occur. It is well known that the fixed-point theorems employed for certain continuous maps may be suitably constructed to prove the existence of such equilibrium.

An interesting feature of the competitive equilibrium is that it attains the best outcomes, not just for individuals, but also for the entire society (economy) in the sense that no one can be made better off from the equilibrium position unless someone is at least made worse off. In other words, the equilibrium position is socially optimal; see Definition 2 for the precise meaning. This is the theoretical interpretation for the celebrated quote:

The socially optimal or efficient states will be realized by the fair competitions *via* the "price mechanism".

Mathematically speaking, the socially optimal states are naturally obtained as solutions of a social optimization problem that maximizes the weighted sum of individual utilities (social welfare function) under the resource constraint. It is well known that the competitive equilibrium is socially optimal (Proposition 1). It can be also proved that each weight profile corresponds to a distinctive optimal state [3]. Therefore, if we know a welfare weight corresponding to the equilibrium, we can also formulate the competitive equilibrium as a solution of the constrained social optimization problem. This point of view was opened up by Negishi [21].

The purpose of this paper is to facilitate Negishi's method with the assistance of recent developments in general equilibrium [3] and optimal control theory [16], see also [11,12], and demonstrate a possible way for extending Negishi's finite dimensional result to infinite dimensional settings. As we explain in the final section, these infinite dimensional spaces naturally arise in many economic applications. We also trust that this analysis introduces mathematicians working in optimal control theory and related areas to general equilibrium theory as an attractive field to which their mathematics is applicable.

The paper is organized as follows. The next section, we present a general equilibrium market model as generally as possible. All of the contents in this section are quite standard; hence, readers familiar with general equilibrium theory can omit this section. In the third section, we formulate our basic problem as a constrained optimization problem as stated above, and prove the existence of the equilibrium (Theorem 1). For some technical reasons, we work within the smooth setting. Certainly this is much stronger than the ordinary continuous setting, but this also makes the analysis simpler and helps to set up the basic framework for considering the competitive equilibrium as an optimal solution.

In the fourth section, we outline the basic idea for extending the finite dimensional model in the next section to infinite dimensional settings. In the infinite dimensional spaces, the social optimization problem will be that of the calculus of variations. To undertake the necessary computations at an elementary level, we utilize the delta function in Dirac [9]. To our knowledge, this is the first time this appears in mathematical economics. The final section concludes.

Abstract Market Model

A market model consists of two categories of theoretical objects: commodities and agents (consumers). The list of commodities available in the market is called the commodity bundle or commodity vector, which is a vector of a Banach space, or more generally, a topological vector space $\boldsymbol{X}$, which we call a commodity space.

Example 1: A commodity space is an ℓ-dimensional Euclidean space R^{ℓ}. Hence a commodity vector is an ordinary ℓ-dimensional vector $\boldsymbol{x} = (x^i)$, $i = 1.... \ell$. A commodity x^i is distinguished from a commodity

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x^j $(i \neq j)$ by its commodity characteristics, for example, its physical specification and the time and location at which it is traded etc.,..

Example 2: A commodity space is ℓ^p space of p-summable sequences for $1 \leq p < +\infty$ [19,24] or the space of all bounded sequences for $p = +\infty$ [5]. A commodity vector is a sequence $\boldsymbol{x} = (x^t)$ and can be interpreted as a stream of commodities along time period t (=1,2...). That x^t is an amount of a commodity available or traded at period t. If we consider the commodity space $L^p(t_0, t_1)$ of p-integrable functions on $[t_0, t_1]$, the trades will be done "instantaneously" at each moment $t \in [t_0, t_1]$.

Example 3: A commodity space is the space of continuous functions $C(K)$ on a compact metric space K [19,24]. In this case, a commodity vector is a continuous function $\boldsymbol{x}(t)$, $t \in \boldsymbol{K}$. The set $\boldsymbol{K}$ is a set of the commodity characteristics and the commodities $\boldsymbol{x}(t)$ are distinguished from each other "continuously". In other words, we can state the two commodities $\boldsymbol{x}(t)$ and $\boldsymbol{x}(s)$ are "near" as their characteristics t and s are near in the metric on K. In terms of economics, the commodities in the space $C(K)$ are "differentiated". We provide some additional remarks on the concept of the differentiated commodities formulated on the space of measures $M(K)$ discussed in [17,13] in the final section.

A consumer is described by its preferences and the commodity bundle initially held (an initial endowment), as explained later. All the examples above are endowed with their order relations, for instance, $\boldsymbol{x} = (x^t) \geq 0 = (0,0,...)$ if and only if $x^t \geq 0$ for all t, $\phi \geq 0$, (constant 0 function) for $\phi(t) \in C(K)$ if and only if $\phi(t) \geq 0$ for all $t \in K$, and so on. Hence, in this paper, the Banach space X is equipped with an order relation denoted as usual by $\geq$ ($\leq$ will be also used). If $\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{x} \neq \boldsymbol{0}$, we write $\boldsymbol{x} > \boldsymbol{0}$. In examples 1 to 3 above, for $\boldsymbol{x} = (x^t)$, $(\boldsymbol{x} = \phi(t))$, we write $\boldsymbol{x} >> \boldsymbol{0}$ if $x^t > 0$, for all t ($\phi(t) > 0$ for all $t \in K$).

Let $\Omega = X_+ = \{\boldsymbol{x} \in X \mid \boldsymbol{x} \geq 0\}$ be the nonnegative orthant of $\boldsymbol{X}$. As each commodity is consumed in a positive (nonnegative) amount, all consumption vectors are potentially contained in the set Ω. Therefore, we refer to this as the consumption set.

Consumers will have some "degree of satisfaction" or "utility" when they consume a bundle $\boldsymbol{x}$. This can be represented by a real valued function called *utility function* $u(\boldsymbol{x})$ defined on the consumption set $\boldsymbol{\Omega}$. For $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\Omega}$, $u(\boldsymbol{x}) \leq u(\boldsymbol{y})$ means that the bundle $\boldsymbol{y}$ is preferred or indifferent to the bundle $\boldsymbol{x}$ by the consumer. $u(\boldsymbol{x}) < u(\boldsymbol{y})$ means that the bundle $\boldsymbol{y}$ is strictly preferred to the bundle $\boldsymbol{x}$. We assume the utility function to be continuous on $\boldsymbol{\Omega}$.

The consumer is also assumed to have a bundle called an initial endowment vector $\omega \in \Omega$, which is owned initially as the consumer's "wealth". This provides the source of money (income) for a consumer to purchase a consumption vector in the market. Finally, we assume that there are finitely many (say m) consumers in the market indexed by $a = 1...m$. The consumer a is characterized by the utility function $u_a(\boldsymbol{x}_a)$ for the consumption bundle $\boldsymbol{x}_a$ and initial endowment ω_a. A $2m$-tuple $(u_a, \omega_a)_{a=1}^m$ is called an *economey.*

A price vector $\boldsymbol{p}$ is a nonnegative and continuous linear functional on the commodity space X that is not equal to the zero functional $\boldsymbol{0}$. It seems economically natural and mathematically convenient that the price functional is an element of the norm dual of the commodity space. Indeed, in all cases in the literature, the price functionals are continuous (or bounded). The value of a commodity bundle $\boldsymbol{x}$ evaluated by a price functional $\boldsymbol{p}$ is denoted by $\boldsymbol{px}$. An m-tuple of the consumption vectors $(\boldsymbol{x}_a)$ is called an allocation. An allocation is said to be feasible if $\sum_{a=1}^m \boldsymbol{x}_a \leq \sum_{a=1}^m \omega_a$. It is called exactly feasible if. $\sum_{a=1}^m \boldsymbol{x}_a = \sum_{a=1}^m \omega_a$. Then we can define the competitive equilibrium of this market in the standard manner.

Definiation 1: An m+1-tuple of consumption bundles and a price vector $((\boldsymbol{x}_a), \boldsymbol{p})$, a =1...m is called the *competitive equilibrium* if and only if the following conditions are satisfied.

(E-1) $\boldsymbol{px}_a \leq \boldsymbol{p}\omega_a$ and $u_a(\boldsymbol{x}_a) \geq u_a(\boldsymbol{x})$ whenever $\boldsymbol{px} \leq \boldsymbol{p}\omega_a$, $a = 1...m$,

(E-2) $\sum_{a=1}^m \boldsymbol{x}_a \leq \sum_{a=1}^m \omega_a$.

The economic meaning of the conditions is clear enough. Condition (E-1) states that consumers maximize their utilities within the budget constraint. Condition (E-2) requires the equilibrium allocation $(\boldsymbol{x}_a)$ to be feasible.

Definiation 2: A feasible allocation $(\boldsymbol{x}_a)$ is called *Pareto optimal* if and only if there exists no other feasible allocation $(\boldsymbol{y}_a)$ such that $u_a(\boldsymbol{y}_a) \geq u_a(\boldsymbol{x}_a)$ for all a, and $u_a(\boldsymbol{y}_a) > u_a(\boldsymbol{x}_a)$ holds for at least for one a.

A feasible allocation is Pareto optimal if it is impossible that everyone can be better off from that allocation with someone strictly better off. The following proposition is easy to understand and almost immediate from Definitions1 and 2.

Proposition 1: Let $((\boldsymbol{x}_a), \boldsymbol{p})$ *be a competitive equilibrium of an economy.* $(u_a, \omega_a)_{a=1}^m$. *Then the allocation* $(\boldsymbol{x}_a)$ *is Pareto optimal.*

proof: Suppose that $(\boldsymbol{x}_1 ... \boldsymbol{x}_m)$ is not *Pareto optimal*. Then there exists a feasible allocation $(\boldsymbol{y}_a)$ such that $u_a(\boldsymbol{y}_a) \geq u_a(\boldsymbol{x}_a)$ for all a and $u_a(\boldsymbol{y}_a) > u_a(\boldsymbol{x}_a)$ holds for at least one a. Then it follows that $\boldsymbol{py}_a \geq \boldsymbol{p}\omega_a$ for all $a \in A$. For a such that $\boldsymbol{p}\omega_a = 0$, the inequality follows trivially. Suppose that $\boldsymbol{p}\omega_a > 0$ and $\boldsymbol{py}_a < \boldsymbol{p}\omega_a$ for some a. Then by the continuity of u_a and $\boldsymbol{p}$, there exists a bundle $\boldsymbol{z} \in \Omega$ which is close enough to $\boldsymbol{y}_a$ such that $u_a(\boldsymbol{y}_a) > u_a(\boldsymbol{z})$ and $\boldsymbol{p}z < \boldsymbol{p}\omega_a$. Therefore, $u_a(\boldsymbol{x}_a) < u_a(\boldsymbol{z})$ and $\boldsymbol{p}z < \boldsymbol{p}\omega_a$, contradicting the condition (E-1) of Definition 1. Furthermore, for a such that $u_a(\boldsymbol{x}_a) < u_a(\boldsymbol{y}_a)$, we have $\boldsymbol{py}_a > \boldsymbol{p}\omega_a$. Summing these inequalities over a, we obtain $\boldsymbol{p}\sum_{a=1}^m y_a > \boldsymbol{p}\sum_{a=1}^m \omega_a$ .Conversely, as the allocatio $(\boldsymbol{y}_1 ... \boldsymbol{y}_m)$ is feasible, we have $\sum_{a=1}^m \boldsymbol{y}_a \leq \sum_{a=1}^m \omega_a$. Given $\boldsymbol{p} > \boldsymbol{0}$, we have $\boldsymbol{p}\sum_{a=1}^m y_a \leq \boldsymbol{p}\sum_{a=1}^m \omega_a$, which is a contradiction.

Let $\omega = \sum_{a=1}^m \omega_a$. The parameter Ω is the amount of the total resource, hence it indicates the 'scale' of the economy. Let $P(\omega)$ be the set of Pareto optimal allocations of the economy with total resource w and we refer to this as the *Pareto set* of the economy $(u_a, \omega_a)_{a=1}^m$. Throughout the paper, we assume that $\omega >> \boldsymbol{0}$.

Finite Dimensional Case: A Fixed Point Problem

In this section, we assume that the commodity space X is R^ℓ and the utility function satisfies the following conditions for every a = 1...m:

(U-1) u_a is twice-continuously differentiable, namely, that of class C^2 on *int* $\Omega = \{\boldsymbol{x} \in \Omega \mid \boldsymbol{x} >> 0\}$,

(U-2) for every sequence $\boldsymbol{x}_n = (x_n^i) \in \text{int}\,\Omega$ such that $x_n^i \to 0$ for some i, it follows that $u_a(x_n^i) \to -\infty$.

We can avoid the "boundary solutions" using the assumption (U-2)

[1]The study of a market model with a continuum of consumers was initiated by Aumann [2], for later development , see Hildenbrand [10] and Suzuki [23].

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Let $Du_a(\boldsymbol{x}) = (\partial_1 u_a(\boldsymbol{x}) \ldots \partial_l u_a(\boldsymbol{x}))$ be the derivatives (tangent map) of u_a at $x \in int\ \Omega$ where

$$\partial_i u_a(\boldsymbol{x}) = \lim_{h\to 0} \frac{u_a(x^1 \ldots x^i + h \ldots x^\ell) - u_a(x^1 \ldots x^i \ldots x^\ell)}{h}$$

and let

$$D^2 u_a(\boldsymbol{x}) = \begin{pmatrix} \partial_1\partial_\ell u_a(\boldsymbol{x}) & \ldots & \partial_1\partial_\ell u_a(\boldsymbol{x}) \\ \ldots & \ldots & \ldots \\ \partial_\ell\partial_1 u_a(\boldsymbol{x}) & \ldots & \partial_\ell\partial_\ell u_a(\boldsymbol{x}) \end{pmatrix}$$

be the second derivative. For every $\boldsymbol{x} \in int\ \Omega$, the Hessian is $D^2u_a(\boldsymbol{x})$ considered to be a linear map from R^ℓ to itself. We also assume for every $a = 1 \ldots m$,

(U-3) u_a is strictly differentiably monotone, i.e., $Du_a(\boldsymbol{x}) >> 0$ for every $\boldsymbol{x} \in int\ \Omega$,

(U-4) u_a is strictly differentiably concave, i.e., $D^2u_a(\boldsymbol{x})$ is a nondegenerate, negative definite bilinear form on R^ℓ.

Let $\Delta = \{\lambda = (\lambda^a) \in \mathbb{R}^m_+ \mid \sum_{\alpha=1}^m \lambda^a = 1\}$ be the unit simplex and for each $\lambda \in int\Delta = \{\lambda \in \Delta \mid \lambda \gg 0\}$ and $\omega \in int\Omega$ consider the following constrained social optimization problem $P(\lambda, \omega)$.

$$P(\lambda,\omega): \text{Maximize} \sum_{a=1}^m \lambda^a u_a(\boldsymbol{x}_a) \text{subject to}$$

$$\boldsymbol{x}_a \in \Omega\ (a = 1 \ldots m), \sum_{a=1}^m \boldsymbol{x}_a \le \omega.$$

For obvious reasons, we call the parameter λ the welfare weight. As the set of feasible allocations $F = \{(\boldsymbol{x}_a) \in \Omega^m \mid \sum_{a=1}^m \boldsymbol{x}_a \le \omega\}$ is compact and convex, the prblem $P(\lambda, \omega)$ has a solution when the utility function $u_a(.)$ is continuous and concave. Balasko [3, Proposition 5.1, p.491] showed that under the assumptions (U-1) to (U-4), the Pareto set $P(\omega)$ is diffeomorphic to $int\Delta$. Let $\boldsymbol{x}_a(\lambda, \omega)$ be the solution. By the monotonicity (U-3), the constraint is binding, or $\sum_{a=1}^m \boldsymbol{x}_a(\lambda,\omega) = \omega$. Balasko also showed $\boldsymbol{x}_a(\lambda, \omega)$ that is smooth at each $(\lambda,\omega) \in int\Delta \times int\ \Omega$[2].

Proposition 1: states that the equilibrium allocation belongs to the Pareto set $P(\omega)$. Therefore, the whole question is to identify the welfare weight corresponding to the equilibrium allocation. We address this problem in this section.

By Corollary 1 in [16, p.219], the solution of the problem $P(\lambda, \omega)$ is a saddle point of the Lagrangian

$$L_{\lambda,\omega}((\boldsymbol{x}_a),\boldsymbol{p}) = \sum_{a=1}^m \lambda^a u_a(\boldsymbol{x}_a) + \boldsymbol{p}(\omega - \sum_{a=1}^m \boldsymbol{x}_a) \tag{1}$$

where $\boldsymbol{p} = (p^i) \in \mathbb{R}^\ell_+$ is the Lagrangian multiplier

Let $\pi(\lambda,\omega) = max\{\sum_{a=1}^m \lambda^a u_a(\boldsymbol{x}_a) \mid \sum_{a=1}^m \boldsymbol{x}_a \le \omega\}$ be the *peramal function* ([16, p.216]). Here we have $\pi(\lambda,\omega) = \sum_{a=1}^m \lambda^a u_a(x_a(\lambda,\omega))$. A remarkable fact is the following, when $\pi(\lambda, \omega)$ is differentiable in ω, the multiplier is determined by $\partial_\omega \pi(\lambda, \omega)$[16, p.222]. Then one has the Lagrangian multiplier $\boldsymbol{p}(\lambda,\omega) = \sum_{a=1}^m \lambda^a Du_a(\boldsymbol{x}_a(\lambda,\omega))\partial_\omega \boldsymbol{x}_a(\lambda,\omega)$.

[2]In fact, Balasko showed the differentiability for λ but not for ω. However, the latter is obvious from his proof.

Let $(\boldsymbol{x}_a(\lambda,\omega), \boldsymbol{p}(\lambda,\omega))$ be the saddle point that satisfies

$$L_{\lambda,\omega}((\boldsymbol{x}_a), \boldsymbol{p}(\lambda,\omega) \le L_{\lambda,\omega}(\boldsymbol{x}_a(\lambda,\omega), \boldsymbol{p}(\lambda,\omega) \le L_{\lambda,\omega}((\boldsymbol{x}_a(\lambda,\omega),\boldsymbol{p}) \tag{2}$$

for every $\boldsymbol{x}_a \in \Omega$ and every $\boldsymbol{p} \in X^*_+$. For each $\lambda \in int\Delta$, define $\hat\lambda = \hat\lambda^a$ by

$$\hat\lambda^a = \frac{max\{0, \lambda^a + \boldsymbol{p}(\lambda,\omega)(\omega_a - \boldsymbol{x}_a(\lambda,\omega))\}}{\sum_{a=1}^m max\{0, \lambda^a + \boldsymbol{p}(\lambda,\omega)(\omega_a - \boldsymbol{x}_a(\lambda,\omega))\}},\ a = 1 \ldots m. \tag{3}$$

Note that $\sum_{a=1}^m \max\{0, \lambda^a + \boldsymbol{p}(\lambda,\omega)(\omega_a - \boldsymbol{x}_a(\lambda,\omega))\} \ne 0$, since otherwise it would follow that $\lambda^a + \boldsymbol{p}(\lambda,\omega)(\omega_a - \boldsymbol{x}_a(\lambda,\omega)) \le 0$ for all $a = 1 \ldots m$. Then would have $1 = \sum_{a=1}^m \lambda^a \le \boldsymbol{p}(\lambda,\omega)\sum_{a=1}^m (\boldsymbol{x}_a(\lambda,\omega) - \omega_a) = 0$, a contradiction. Hence $\hat\lambda = (\hat\lambda^a) \in \Delta$ is well defined.

Let $\Phi: int\Delta \to \Delta$ be defined by $\Phi(\lambda) = \hat\lambda$. As $\boldsymbol{x}_a(\lambda,\omega)$ and $\boldsymbol{p}(\lambda,\omega)$ are continious on $int\Delta$ and Δ is compact, we can extended Φ continiously to Δ. Then by Brower's fixed-point theorem, there exists a fixed point

$$\hat\lambda = \Phi(\hat\lambda).$$

Let $\boldsymbol{x}_a(\hat\lambda,\omega) = \hat{\boldsymbol{x}}_a$ and $\boldsymbol{p}(\hat\lambda,\omega) = \hat{\boldsymbol{p}}$. In the following, we will show that the fixed point corresponds to a competitive equilibrium $((\hat{\boldsymbol{x}}_a), \hat{\boldsymbol{p}})$.

We first note that the equilibrium price is strictly positive, or $\hat{\boldsymbol{p}} = (\hat p) \gg \boldsymbol{0}$. If not, $\hat p^j = 0$ for some j. Define a new allocation $(\boldsymbol{y}_a)_{a=1}^m = \left((y^i_a)_{a=1}^m\right) \in \Omega$ by $y^i_a = \hat x^i_a$ for $i \ne j$ and $y^i_a = \hat x^i_a + \epsilon$ for some $\epsilon > 0$. Then $L_{\hat\lambda,\omega}((\boldsymbol{y}_a), \hat{\boldsymbol{p}}) > L_{\hat\lambda,\omega}((\boldsymbol{x}_a), \hat{\boldsymbol{p}})$, contradicting the first inequality of (2).

Then we have $\hat\lambda_{a'} > 0$ for a' such that $\omega_{a'} > \boldsymbol{0}$. If not, it follows from (3) that $0 < \hat{\boldsymbol{p}}\omega_{a'} \le \hat{\boldsymbol{p}}\hat{\boldsymbol{x}}_{a'}$, hence $\boldsymbol{x}_{a'} > \boldsymbol{0}$. Since $\hat\lambda \in \Delta, \hat\lambda_b > 0$ for some b. Define a new collection $(z_a)_{a=1}^m$ by $z_{a'} = \boldsymbol{0}$[3] $z_b = \hat{\boldsymbol{x}}_b + \hat{\boldsymbol{x}}_{a'}$, and $z_a = \hat{\boldsymbol{x}}_a$ for $a \ne a$, b. Then $\sum_{a=1}^m z_a \le w$ and $\sum_{a=1}^m \hat\lambda^a u_a(\hat{\boldsymbol{x}}_a) < \sum_{a=1}^m \hat\lambda^a u_a(z_a)$ contradicting the first inequality of the saddle point property (2). Similarly, we show that for a with $\omega_a = \boldsymbol{0}$, it follow that $\hat\lambda^a = 0$ and $\hat{\boldsymbol{x}}_a = \boldsymbol{0}$. Then we have form (3) that $\hat\lambda^a = \hat\lambda^a + \hat{\boldsymbol{p}}\omega_a - \hat{\boldsymbol{p}}\hat{\boldsymbol{x}}_a$ hence, the budget constraints for all consumers are met, namely that $\hat{\boldsymbol{p}}\hat{\boldsymbol{x}}_a = \hat{\boldsymbol{p}}\omega_a$, a= *1…m*.

When $\omega_a = \boldsymbol{0}$, the set of consumption bundles satisfying the budget constraint is a singleton $\{\boldsymbol{0}\}$, given $\hat{\boldsymbol{p}} \gg 0$.Therefore, the bundle **0** maximizes the utility trivially, and hence satisfies the condition (E-1). Now suppose that $\omega_a > \boldsymbol{0}$ and take any vector $\boldsymbol{x}$ with $\hat{\boldsymbol{p}}\boldsymbol{x} \le \hat{\boldsymbol{p}}\omega_a$. Setting $x_b = \hat x_b$ for $b \ne a$, it follows from the first inequality of (2) that $\hat\lambda^a u_a(\boldsymbol{x}) \le \hat\lambda^a u_a(\boldsymbol{x}) + \hat{\boldsymbol{x}}(\omega_a - \boldsymbol{x}) \le \hat\lambda^a u_a(\hat{\boldsymbol{x}}_a)$, hence the equilibrium condition (E-2) is also met (with exact equality). We have thus prooved:

Theorem 1. *There exists a competitive equilibrium.*

We now provide a simple example for which we can easily compute the equilibrium.

Example 4: A commodity vector is denoted by $\boldsymbol{x} = (x^i)$, $i = 1 \ldots \ell$. The consumer a (=*1…m*). Note that this utility function[3] $u_a(x_a) = \sum_{i=1}^\ell \beta^i_a \log x^i_a$ where the constants β^i_a's satisfy $\beta^i_a > 0$, for all $i = 1 \ldots \ell$ and $\sum_{i=1}^\ell \beta^i_a = 1$, all a =*1…m*). Note that thia utility function satisfies the conditions U-1 and U-4. Each consumer is assumed to be consumed to own the same amount of commodities as his/her initial endowment $\omega_a = (e_a \ldots e_a)$ with $e_a > 0$. Let $= \left(\sum_{a=1}^m e_a \ldots \sum_{a=1}^m e_a\right) = (e \ldots e)$ be the total endowment (resource) of the economy.

[3]In the economic literature, this is the Cobb--Douglas utility function.

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The Lagrangian for this example is written by

$$L_{\lambda,\omega}\left((\boldsymbol{x}_a),\boldsymbol{p}\right)=\sum_{a=1}^{m}\lambda^a\sum_{i=1}^{\ell}\beta_a^i\log x_a^i+\boldsymbol{p}\left(\omega-\sum_{a=1}^{m}\boldsymbol{x}_a\right). \quad (4)$$

Setting $(\lambda^a)=(\hat{\lambda}^a)$ in the first-order conditions (FOCs) for $L_{\lambda,\omega}$, evaluated at $(\hat{\boldsymbol{x}}_a,\hat{\boldsymbol{p}})$, we obtained

$$\frac{\partial L_{\hat{\lambda},\omega}}{\partial x_a^i}=\hat{\lambda}_a\beta_a/\hat{x}_a^i-\hat{p}^i=0, i=1\ldots\ell, a=1\ldots m \quad (5)$$

Summing (5) over i and using the budget condition $\sum_{i=1}^{\ell}p^ix_a^i=e_a\sum_{i=1}^{\ell}p^i$ and $\sum_{i=1}^{\ell}\beta_a^i=1$, we have $\hat{\lambda}^a=e_a\sum_{i=1}^{\ell}p^i, a=1\ldots m$. Summing this over a with the help of $\sum_{a=1}^{m}\hat{\lambda}^a=1$, the welfare weights are determined by

$$\hat{\lambda}^a=\frac{e_a}{e} \quad (6)$$

and we obtained from (5) and (6) that

$$\hat{p}^i=\frac{\sum_{a=1}^{m}e_a\beta_a^i}{e^2}, i=1\ldots\ell, \quad (7)$$

$$\hat{x}_a^i=\frac{ee_a\beta_a^i}{\sum_{a=1}^{m}e_a\beta_a^i},\ i=1\ldots\ell,\ a=1\ldots m. \quad (8)$$

Infinite Dimensional Case: A Variational Problem

In this section, we consider the market model in which the commodities are indexed by time period $t \in T$ (intertemporal resource allocation model). In what follows, we assume that T is a compact interval $T=[t_0,t_1]$ with Lebesgue measure τ. The (nonnegative) amount of the commodity t is denoted by $x(t)$, $y(t)$ and so on. Hence, the commodity bundle is a function defined on T. We assume that the commodity space to be the space of essentially bounded functions $L^\infty[t_0,t_1]=\{f:T\to\mathbb{R} \mid \|f\|_\infty<+\infty\}$, where $\|f\|_\infty=\sup\{r\ge 0 \mid \tau(\{t\in T \mid |f(t)\ge r\})>0\}$. A pleasant property of L^∞ is that the interior of $\Omega=L_+^\infty$ with respect to the norm topology is nonempty. It is well known that the norm dual of L^∞ is the set of bounded additive set functions (finitely additive measures) on $(T, B(T))$ which is denoted by $ba(T)$, where $B(T)$ is the borel σ-algebra of T. The value of a liner functional $\pi\in ba(T)$ at $f\in L^\infty(T)$ is denoted by $\pi f=\int_T f(t)\,d\pi$.

As before, there exist m consumers indexed by a(=1,...m). Consumer a has the separable utility function[4],

$$U_a(x_a(t))=\int_T\beta_a(s)u_a(x_a(s))ds,\ a=1\ldots m,$$

Where $\beta_a(t)>0$ for all $t\in T$ and $\int_T\beta_a(s)ds=1$, a = 1 . . . m, and $u_a(x)$ is a real valued function on $\mathbb{R}_{++}$ that satisfies the conditions (U-1) to (U-4) in the previous section. We assume that the consumer a is endowed with $\omega_a(t)(>0)$ amount of the commodity t as his/her initial endowment. Set $\omega(t)=\sum_{a=1}^{m}\omega_a(t)$.

In the following example, we illustrate our basic computational tool (Dirac's delta function) as used throughout this section.

Example 5: There exists one consumer m=1 with utility function $u(x(t))=\int_T\beta(s)\log x(s)ds$ such that $\beta(t)\ge 0$ for all $t\in T$ and $\int_T\beta(s)ds=1$. His/her initial endowment $\omega(t)=\omega\ (>0)$ is assumed to be constant, hence $\omega(t)=\omega$ is also the total endowment (resource) of the economy at each t.

[4]More precisely, the utility function is not a function but a (nonlinear) functional that is a map assigning a real value to the function (not a number) $x(t)$.

To elucidate the functional calculus in an elementary way, Dirac's cerebrated *delta function* will be helpful. It is a "function" $\delta(t)$ on $\mathbb{R}$ that is defined by

$$\delta(t)=\begin{cases}0 & for\ t\neq 0\\ +\infty & for\ t=0\end{cases}$$

and assumed to satisfy $\int_{-\infty}^{+\infty}\delta(s)ds=1$. From this and the definition, we obtain that $\int_T g(s)\delta(t-s)ds=g(t)$ for any function[5] $g(t)$([9, p.59]).

Let f(x) differentiable function. we define

$$\frac{df(x(s))}{dx(t)}\equiv\lim_{h\to 0}\frac{f(x(s))+h\delta(t-s)-f(x(s))}{h}=f'(x(s))\delta(t-s).$$

This is a fundamental mathematical formula, which we use throughout the note. Then we can differentiate $\int_T f(x(s))ds$ with respect to $x(t)$:

$$\frac{d}{dx(t)}\int_t f(x(s))ds=\int_T\frac{df(x(s))}{dx(t)}ds=\int_T f'(x(s))\delta(t-s)ds=f'(x(t)).$$

The consumer maximizes the utility function(al)

$$u(x(t))=\int_T\beta(s)\log x(s)ds$$

subject to the budget constraint $\int_T p(s)x(s)ds=\int_T p(s)\omega(s)ds=\omega$ where we have assumed that the equilibrium price vector $p(t)$ can be taken form $L^1(T)$, and nomalized $\int_T p(s)ds=1$ (this assumption will removed in what follows). Mathematically speaking, this is to solve a constrained variational problem. To this end, we differentiate the constrained Lagrangian with the multiplier μ.

$$L(x(t),\mu)=\int_T\beta(s)\log x(s)ds+\mu\left(\int_T p(s)(\omega(s)-x(s)ds\right)$$

in $x(t)$ and obtained the FOC

$$\frac{dL(x(t),\mu)}{dx(t)}=\frac{\beta(t)}{x(t)}-\mu p(t)=0\cdot$$

It follows from the FOC that $\int_T\beta(s)ds-\mu\int_T p(s)x(s)ds=0$, hence $\mu=1/\omega$. Therefore, the demand function for $x(t)=\beta(t)\omega/p(t)$. The equlibirium price $\hat{p}(t)$ can be obtained from the market condition $\hat{x}(t)=\omega, \hat{p}(t)=\beta(t)$.

The above calculation of the equilibrium for the one-consumer economy is straightforward. Our economic problem is to examine (and compute for a special case, see Example 6 below) the competitive equilibrium of the multiconsumer economy in the similar way as that for the finite dimensional case.

Now consider the constrained variational problem, which is completely analogous with the constrained optimization problem $P(\lambda,\omega)$ in the previous section:

$$\Pi(\lambda,\omega):\text{Maximize}\sum_{a=1}^{m}\lambda_a\int_T\beta_a(s)u_a(x_a(s))ds$$

$$\text{subject to}\ \sum_{a=1}^{m}x_a(t)\le\omega(t),\ t\in T,$$

where λ^1. . . λ^m are the welfare weights of consumer satisfying $\lambda_a\ge 0$ and $\sum_{a=1}^{m}\lambda^a=1$. By Corollary 1 in [16, p.219], the solution of the problem is a saddle point of the Lagrangian

$$L_{\lambda,\omega(t)}((x_a(t)),\pi)=\sum_{a=1}^{m}\lambda_a\int_T\beta_a(s)u_a(x_a(s))ds$$
$$+\pi(\omega(s)-\sum_{a=1}^{m}x_a(s)), \quad (9)$$

where $\pi\in ba(T)$ is the multiplier (a fortiori it will be equlibirium price vector) and the condition $int\Omega\neq\emptyset$ is used.

[5] We set aside the problem to specify an appropriate condition here.

Let the saddle point be $(\hat{x}_a(t), \hat{\pi})$. Note that the saddle point still depends on (λ_a). The competitive equilibrium should be the fixed point of the map

$\Phi((\lambda^a)) = ((\hat{\lambda}^a))$,

where $\hat{\lambda}^a$ is defined similarly as for the finite dimensional case:

$$\hat{\lambda}^a = \frac{max\{0, \lambda^a + \hat{\pi}(\omega^a(s) - \hat{x}_a(s))\}}{\sum_{a=1}^m max\{0, \lambda^a + \hat{\pi}(\omega^a(s) - \hat{x}^a(s))\}}, \ a = 1...m. \qquad (10)$$

Suppose that the fixed point $\hat{\lambda}$ of Φ exists. We can show that the saddle point $((\hat{x}_a(t)), \hat{\pi})$ of $L_{\hat{\lambda},\omega}$ (or the solution of $\Pi(\hat{\lambda},\omega)$ that satisfies

$$L_{\hat{\lambda},\omega}((x_a(t)),\hat{\pi}) \le L_{\hat{\lambda},\omega}((\hat{x}_a(t)),\hat{\pi}) \le L_{\hat{\lambda},\omega}((\hat{x}_a(t)),\pi)$$

$$\text{for all } x_a(t), \pi$$

is a competitive equilibrium exactly in the same way as in the previous section. By Theorem 2 in [5, p.523] , the equilibrium price must belong to $L^1(T)$. Hence we can write $\hat{\pi} = \hat{p}(t)$ and $\hat{\pi}x_a(s) = \int_T x_a(s)d\hat{\pi} = \int_T \hat{p}(s)x_a(s)ds$. In particular, the budget condition is written as $((\hat{x}_a(t)), \hat{p}(t))$ and $(\hat{\lambda}_a)$.

$$\int_T \hat{p}(s)\hat{x}_a(s)ds = \int_T \hat{p}(s)\omega_a(s)ds$$

for each a = 1. . . m. The equlibirum are characterized by FOC's evaluated at

$$\frac{\partial L_{\hat{\lambda},\omega(t)}((\hat{x}_a(t)),\hat{p}(t))}{\partial x_a(t)} = \hat{\lambda}^a\beta_a(t)u'(\hat{x}_a(t)) - \hat{p}(t) = 0,$$

$$a = 1....m, \qquad (11)$$

$$\frac{\partial L_{\hat{\lambda},\omega(t)}((\hat{x}_a(t)),\hat{p}(t))}{\partial p(t)} = \omega(t) - \sum_{a=1}^m \hat{x}_a(t) = 0, \qquad (12)$$

From these conditions, we can explicitly compute the equilibrium for the case of log-linear utilities.

Example 6: Let the utility function of the consumer a be the Cobb-Douglas form $U_a(x(t)) = \int_T \beta(s)\log x_a(s)ds$ such that $\beta_a(t) \ge 0$ for all $t \in T$ and $\int_t \beta_a(s)ds = 1$, and the endowment bundle is a constant function $\omega_a(t) = \omega_a \ge 0$ for all t as in Example 5. Note that the case $\omega_a = 0$ for some a is not excluded. Let $\omega = \sum_{a=1}^m \omega_a$. We assume ω > 0. The following calculation will proceed similarly as in Example 4.

The FOC (11) now reduces to

$$\frac{\partial L_{\hat{\lambda}}}{\partial x_a(t)} = \hat{\lambda}^a\beta(t)/\hat{x}_a(t) - \hat{p}(t) = 0, a = 1...m. \qquad (13)$$

Integrating (13) and using the budget equation $\int_T \hat{p}(s)\hat{x}_a(s)ds = \omega_a\int_T \hat{p}(s)ds$ and $\sum_{a=1}^m \hat{\lambda}^a = 1$, we have[6]

$$\hat{\lambda}^a = \frac{\omega_a}{\omega}, \ a = 1....m. \qquad (14)$$

Summing (13) over a with the help of (14), it follows that

$$\hat{p}(t) = \frac{\sum_{a=1}^m \omega_a\beta_a(t)}{\omega^2}, \ t \in T, \qquad (15)$$

$$\hat{x}_a(t) = \frac{\omega\omega_a\beta_a(t)}{\sum_{a=1}^m \omega_a\beta_a(t)}, \ t \in T, \ a = 1...m. \qquad (16)$$

Therefore, we have obtained:

Proposition 2: *For the economy with log-linear utilities and constant endowment bundles, the competitive equilibrium is given by (15) and (16).*

Concluding Remarks

The most important lesson of Negishi [21] is that the Lagrangian multiplier is nothing but an equilibrium price vector when the competitive equilibrium is characterized as a saddle point of the Lagrangian[7] Indeed, all of the information concerning equilibrium is included in the Lagrangian[8] (1) or (9) and the fixed point map (3) or (10). We have thus obtained a simple and elegant scheme, which describes the equilibrium state of a market generated by complex interactions between consumers and commodities.

The formulation of market equilibrium in the present paper opens up a close relationship between economic theory and optimal control theory, and we expect that our understanding of the former will be advanced by development of the latter. In particular, further developments in constrained optimal theory with Lagrangian multipliers on infinite dimensional spaces are earnestly desired. For instance, consider the question of the non-empty interior of Ω. This comes from the condition assumed in the Hahn-Banach separation theorem which requires that the at least one of separating convex sets has a nonempty norm interior. The same problem occurs in general equilibrium theory, and the problem has been handled by introducing the 'proper preferences' [19,24]. This idea and technique might provide some help for the mathematics of optimization problems. We would like to stress that economists and mathematicians can collaborate naturally in this area.

Given the commodities are distinguished from each other by their characteristics, the infinite dimensional commodity space naturally arises in economic theory. Bewley [5] was one of the first studies in which the commodity space is L∞(T), the space of essentially bounded functions on the interval T = [t0, t1] as a model of intertemporal trades, and proved the existence of equilibria under very general conditions. Mas-Colell [17] introduced the idea of commodity differentiation into an abstract concept that treats the differentiated commodity as a measure (distribution) m on a compact metric space K. This means that for each measurable set B, m(B) is a real value interpreted as an amount of the commodity m, that contains a portion $B \subset K$ of the characteristics. In this definition, the commodities are set functions, not simply functions, as in our paper. Jones [13] simplified Mas-Colell's proof and Khan and Suzuki [15] further elaborated upon their work.

Mas-Colell [19] and Yannelis-Zame [24] generalized those theorems of Bewley and MasColell to more abstract spaces such asBanach lattices or topological vector lattices. However, there is an important distinction between [5,13, 15,17] and [19,24]. In the former group of works, the commodity spaces are the dual spaces (recall that $L^\infty(T) = L1(T)^*$, and $M(K) = C(K)^*$ and they worked with the weak* topology. Consequently the price spaces are the predual spaces $L1$(T) for L∞(T), and C(K) for M(K). In [19,24], the price spaces are simply the (norm) dual spaces. An appropriate topology in each case should be determined through both mathematical and economic considerations.

[6]Note that the price cannot be normalized independently of the normalization of λ_a. If we normalized the prices as $\int_t \hat{p}(s)ds = 1$ as in Example 5. then we would have $\lambda_a = \omega_a$.

[7]This point was lost form the concept of G-equilibrium in Balasko [3,4]. Neverthless Balasko's equilibrium concept is appropriate for the analysis of local uniqueness and stability not discussed in this paper. For these important issues, readers should refer to [7,8,18].

[8]Physicists might well be reminded that the Lagrangian of the standard model, e.g., Kane [14] , includes all information for the interactions between the elementary particles.

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Competing Interests

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