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Evaluation of Ray-Path Integrals in Geometrical Optics

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Abstract

A brief summary of the physical context to this paper is provided, and the deviation angle undergone by an incident ray after k internal reflections inside a transparent unit sphere is formulated. For radially inhomogeneous spheres (in particular) this angle is related to a ray-path integral; an improper integral for which there are relatively few known exact analytical forms, even for simple refractive index profiles *n(r)*. Thus for a linear profile the integral is a combination of incomplete elliptic integrals of the first and third kinds (though not all are as complicated as this). The ray-path integral is evaluated for ten different *n(r)* profiles, many of which have not been provided elsewhere. In the appendix a *mirage theorem* is proved for horizontally stratified media. This illustrates the more general principle in geometrical optics, namely that a ray path is always concave towards regions of higher refractive index.

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Introduction: Snell's law of refraction

Solutions of spherical scattering problems have practical applications in chemistry, physics, microbiology, meteorology, radar, astronomy, and other fields. Many phenomena that we experience every day are related to the scattering of plane electromagnetic waves. Sound and light waves are scattered around objects that enable us to hear the sound and be illuminated by the light. The scattering of plane electromagnetic waves provides an explanation of why the sky is blue and how a rainbow is formed. There are several other reasons why it is important to have a deep understanding of electromagnetic scattering by radially inhomogeneous media. Methods that are employed in this area can be very useful in exploring the combustion of liquid hydrocarbons, the injection of sprays in high pressure environments, as well as the spraying and drying techniques utilized in the food, agricultural, and pharmaceutical industries. Another example where the scattering of electromagnetic waves by a radially inhomogeneous sphere is used is in biological studies to detect blood and bacteria cells. Medical imaging uses the scattering of plane electromagnetic waves to identify and diagnose a range of health-related issues. Electromagnetic scattering is also utilized in geophysical exploration to identify a new deposit of a certain resource. Another example of the importance of electromagnetic scattering is in the area of nondestructive testing of artifacts without causing damage to the environment or other objects. The scattering of electromagnetic plane waves by a radially inhomogeneous sphere is a vast field with many practical and research applications [2].

When rays propagate in inhomogeneous media, a condition that is sometimes placed on the applicability of geometrical optics is that the refractive index profile in the medium must be slowly varying [7]. Plasmas are an excellent example of media, which in limiting cases, may exhibit poles, zeros or both in the refractive index. For example, a cylindrically confined laboratory plasma may possess a resonance, cutoff, or both at some finite radii. If the frequency of the incident wave is much greater than the collision frequency of the plasma, as it occurs when the plasma is probed by a laser beam, then the squared refractive index is essentially a real quantity, and it is infinite at a resonance and zero at a cutoff (this is equivalent to saying that collisions are neglected) [7]. The current and future applications of plasmas provide reasons why the study of singular refractive index profiles is of considerable value.

To set the scene, so to speak, some basic underlying physical principles are stated here. As is well-known to students of elementary physics, Snell's law of refraction defines the relationship between the refractive index of a medium and the direction of the light ray at that point. For continuously varying media, if $\theta(\mathbf{r})$ is the angle made by the ray with some reference direction (depending on the coordinate system used) and $n(\mathbf{r})$ is the refractive index:

$$n(\mathbf{r}) \sin \theta(\mathbf{r}) = constant. \tag{1}$$

For the special case of a boundary between two uniform media with refractive indices n_1 and n_2 equation (1) reduces to

$$n_1 \sin \theta_1 = n_2 \sin \theta_2. \tag{2}$$

In practice, in physics texts at least, this is written in terms of the angle of incidence (i), the angle of refraction ($\tilde{r}$) and the relative refractive index $\theta_1 = i$, $\theta_2 = \tilde{r}$ and $n = n_2/n_1$:

$$\sin i = n \sin\tilde{r}. \tag{3}$$

Note that the standard notation for the angle of refraction (r) is here being replaced by ($\tilde{r}$) to avoid confusion with the radial variable.

Using these results it can be shown that the *deviation angle* undergone by an incident ray after k internal reflections inside a transparent unit sphere is, for constant refractive index n

$$\Theta_k(i) = 2i + k\pi - 2(k+1)\tilde{r}(i)$$
$$\equiv 2i + k\pi - 2(k+1)\arcsin\left(\frac{\sin i}{n}\right). \tag{4}$$

This formula arises because in addition to the deviation (i-$\tilde{r}(i)$) at both the entry and exit points of the sphere, each reflection induces a deviation of (π-2$\tilde{r}(i)$) radians. This formula (along with its extension to radially inhomogeneous spheres) is of importance in the field of geometric optics, especially as applied to 'rainbows' and higher-order bows, as well as so-called 'zero-order' bows (which exist only in nonhomogeneous media) [1-3]. Such bows exist at extrema of $\Theta_k(i)$, i.e. when $\Theta'_k(i) = 0$.

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Spherically Symmetric Media

In a spherically symmetric medium with refractive index $n(r)$ it is readily shown that each ray path satisfies the following equation [2,4]

$$rn(r)\sin\phi = constant, \tag{5}$$

where ϕ is the angle between the radius vector r and the tangent to the ray at that point, where $r = |\mathbf{r}|$. The above expression may be thought of as the optical analogue of the conservation of angular momentum for a particle moving under the action of a central force. The result, known as Bouguer's formula (for *Pierre Bouguer*,1698-1758), implies that all the ray paths $r(\theta)$ are curves lying in planes through the origin where θ is the polar angle. By using elementary differential geometry, we can establish that

$$\sin\phi = r(\theta)\left[r^2(\theta) + \left(\frac{dr}{d\theta}\right)^2\right]^{-1/2}. \tag{6}$$

In this paper we examine certain integrals that arise in the mathematical theory of ray scattering by radially inhomogeneous spheres. Instead of i, θ_i will refer to the angle of incidence for the incoming ray, r is the radial distance within a unit sphere (since any sphere is easily scaled to r=1 by a redefinition of the original radius), and $\Theta_k(\theta_i)$is the total angle of deviation undergone by the ray from its original direction after k internal reflections. Thus the subscripts 0 and 1 are used to distinguish between the deviations of the exiting ray for direct transmission and the primary bow, respectively. It is a well-known result that the curvature of the ray path is towards regions of higher refractive index n – a consequence of Snell's law of refraction in continuously-varying media. This is a consequence of a general principle (the *mirage theorem*), a special but important case of which is discussed in the Appendix. Thus if $n'(r) < 0$ an incoming ray is concave towards the origin (i.e. it curves *around* it); if $n'(r) > 0$, it is concave away from it (or curves *away* from it). If $2\Delta(\theta_i)$ is the angle subtended at the center of the sphere by the ray path inside the sphere, then for *each* internal reflection the ray undergoes an additional deviation of $2\Delta(\theta_i)$. We shall refer to $2\Delta(\theta_i)$ as the basis *angle* for the ray path inside the sphere. Hence the total deviation for k internal reflections is

$$\Theta_k(\theta_i) = 2\theta_i - \pi + 2(k+1)\Delta(\theta_i). \tag{7}$$

The quantity $\Delta(\theta_i)$ is an improper definite integral to be defined below for a unit sphere. Apart from a few specific $n(r)$ profiles, analytic expressions for $\Delta(\theta_i)$ are difficult to obtain. In this paper $\Delta(\theta_i)$ is evaluated for ten such cases. It is interesting to note that even for simple $n(r)$ profiles (such as the linear profile) the evaluation of the integral is quite challenging, in fact a great deal of elementary (and on occasion quite sophisticated) analysis is required to obtain explicit closed-form expressions for $\Delta(\theta_i)$.

From equations (5) and (6) the formula for $\Delta(\theta_i)$ is

$$\Delta(\theta_i) = \sin\theta_i \int_{r_c(\theta_i)}^{1} \frac{dr}{r\left[r^2 n^2(r) - \sin^2\theta_i\right]^{1/2}}, \tag{8}$$

where the lower limit $r_c(\theta_i)$is the point at which the integrand is singular, being the largest solution of

$$r_c(\theta_i)\, n(r_c(\theta_i)) = \sin\theta_i \tag{9}$$

provided $0<r_c<1$.The quantity $r_c(\theta_i)$ is the radial point of closest approach to the center of the sphere which is sometimes called the *turning point*. As noted earlier, for a kth-order bow to exist at some critical angle of incidence θ_{i_c} it is necessary and sufficient that

$$\Theta_k'(\theta_{i_c}) = 0. \tag{10}$$

Equation (8) will be evaluated for ten distinct refractive index profiles. That result can then be employed in equation (7), and the resulting equation utilized to impose conditions on the refractive index $n(r)$ that permits a bow of any order to exist. This procedure will be applied only to those refractive index profiles for which $\Theta_k'(\theta_{i_c})$is readily obtainable algebraically (though in some cases not without difficulty). It should be noted at the outset that some of the profiles discussed below can be singular within the unit sphere, depending on the particular choice of parameters. Nevertheless we will here choose the relevant parameters such that the profiles are non-singular within the refracting sphere (though it is not always necessary to do so [5]).

Refractive Index Profiles

Profile 1

The refractive index profile considered is

$$n(r) = \frac{2n_1 r^{1/c-1}}{1+r^{2/c}},\quad n_1 = n(1), \tag{11}$$

where c is a positive real constant. This profile is singular at $r = 0$ for $c > 1$. It is also singular at $r = 0$ for $c < 0$. In order to avoid such singular behavior it will be assumed that $0 < c \le 1$. When $c = 1$, the refractive index profile in equation (11) results in the well-known Maxwell's fish-eye profile. *Maxwell's fish-eye* profile was studied by James Clerk Maxwell in 1854, but without the spherical mirror. The reason why it is called the fish-eye profile is because Maxwell thought of this profile by considering the crystalline lens in fish. The fish-eye mirror makes a perfect lens, but it is a rather peculiar lens that contains both the object and the image inside the optical medium. The fish-eye mirror could transfer embedded images with details significantly smaller than the wavelength of light over distances much larger than the wavelength, a useful feature for nanolithography [14].

If we define the quantity

$$\hat{a} \equiv \frac{\sin\theta_i}{2n_1} < \frac{1}{2}$$

it follows that

$$r_c^{1/c} = \frac{1 \pm \sqrt{1-4\hat{a}^2}}{2\hat{a}}.$$

It is readily demonstrated that the '+' root corresponds to $r_c>1$, i.e. outside the unit sphere, so it follows that

$$r_c(\theta) = \left[\frac{1-\sqrt{1-4\hat{a}^2}}{2\hat{a}}\right]^c \tag{12}$$

from which the following remarkably simple result is obtained,

$$\Delta(\theta_i) = \frac{c}{2}\pi. \tag{13}$$

Hence, for the refractive index given by profile 1 we find that

$$\Theta_k(\theta_i) = 2\theta_i - \pi + 2(k+1)\Delta(\theta_i) = 2\theta_i - \pi + (k+1)c\pi$$
$$= \pi\left[(k+1)c - 1\right] + 2\theta_i. \tag{14}$$

Note that for this profile, $\Theta_k'(\theta_i) = 2$ for any value of i. Thus, *no bow exists for any order k*, a quite remarkable conclusion given the mathematical complexity of the profile (11) as compared with a constant refractive index (for which bows do exist).

Profile 2

Consider next the refractive index profile

$$n(r) = n(0)\left(1-\frac{r^2}{L^2}\right)^{1/2} \equiv n_0\left(1-\frac{r^2}{L^2}\right)^{1/2}. \tag{15}$$

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This is known (not surprisingly) as the parabolic refractive index profile, and was examined by [11,12]. It has been used in studies of cylindrical optical fibers where r is the distance from the optical fiber axis. L^2 is a constant that can be determined from knowledge of the core and surface refractive indices. Let $n(1) \equiv n_1 > 1$. Then we find that

$$L^2 = \frac{n_0^2}{n_0^2 - n_1^2}.$$

Let $K = \sin\theta_i$. Using equation (9) it is found that

$$r_c^2 = \frac{n_0^2 \pm \left[n_0^4 - 4\left(n_0^2 - n_1^2\right)K^2\right]^{1/2}}{2\left(n_0^2 - n_1^2\right)}.$$

For the unit sphere the radial point of closest approach is bounded by $0 \le r_c < 1, \quad r_c \in R$.

Further restrictions are imposed on $r_c(\theta_i)$ that are dependent on the values of n_0 and n_1. Two cases will be summarized here.

Case 1 ($1 < n_1 < n_0$)

This corresponds to the refractive index decreasing monotonically from the center. It can be shown that

$$r_c = \left\{\frac{n_0^2 - \left[n_0^4 - 4\left(n_0^2 - n_1^2\right)K^2\right]^{1/2}}{2\left(n_0^2 - n_1^2\right)}\right\}^{1/2}, \tag{16}$$

subject to the constraint

$$n_0^4 > 4\left(n_0^2 - n_1^2\right). \tag{17}$$

Case 2 ($1 < n_0 < n_1$)

This corresponds to the refractive index increasing monotonically from the center. It can be shown that the very same result (16) holds in this second case also. Consequently the basis for the interior angle of deviation is

$$\Delta(\theta_i) = \frac{1}{2}\left\{\arcsin\left[\frac{n_0^2 - 2K^2}{\left[n_0^4 - 4\left(n_0^2 - n_1^2\right)K^2\right]^{1/2}}\right] + \frac{\pi}{2}\right\}. \tag{18}$$

Therefore, for the refractive index profile 2, we find

$$\Theta_k(\theta_i) = 2\theta_i - \pi + 2(k+1)\Delta(\theta_i) \tag{19}$$

$$= 2\theta_i + (k-1)\frac{\pi}{2} + (k+1)\arcsin\left[\frac{n_0^2 - 2K^2}{\left[n_0^4 - 4\left(n_0^2 - n_1^2\right)K^2\right]^{1/2}}\right].$$

As in all these profiles, conditions for the existence of kth-order bows can be found by seeking extrema of $\Theta_k(\theta_i)$.

Profile 3

Consider next the profile

$$n(r) = a - br^2 \tag{20}$$

where a and b are real constants. This profile in particular has been used to extend the Airy rainbow theory to nonuniform spheres [1, 10]. It transpires that the critical radius r_c satisfies the cubic equation

$$br^3 - ar + K = 0 \tag{21}$$

so care must be taken in evaluating the three roots (not all of which will be physically interesting). These roots will be denoted by r_1, r_2, and r_c where $r_c < \min(r_1, r_2)$ if r_1 and r_2 are real. After a great deal of algebra the basis internal angle $\Delta(\theta_i)$ can be expressed succinctly in terms of $\Pi(\phi, \alpha^2, p)$ the *incomplete elliptic integral of the third kind*, where

$$\Pi(\phi, \alpha^2, p) = \int_0^{\sin\phi} \frac{dt}{\left(1 - \alpha^2 t^2\right)\sqrt{\left(1 - t^2\right)\left(1 - p^2 t^2\right)}} \tag{22}$$

and

$$\sin\phi = \left(\frac{r_c^2 - r^2}{r_c^2 - r_2^2}\right)^{1/2}, \quad p = \left(\frac{r_c^2 - r_2^2}{r_c^2 - r_1^2}\right)^{1/2} \; and \; \alpha^2 = 1 - \frac{r_2^2}{r_c^2}. \tag{23}$$

From equation (8) we obtain

$$\Delta(\theta_i) = \frac{Kj}{|b| r_c^2 \sqrt{r_c^2 - r_1^2}} \Pi(\phi, \alpha^2, p). \tag{24}$$

Therefore we can determine that

$$\Theta_k(\theta_i) = 2\theta_i - \pi + 2(k+1)\frac{Kj}{|b| r_c^2 \sqrt{r_c^2 \quad r^2}} \Pi(\phi, \alpha \quad, p). \tag{25}$$

As can be seen from this expression, the calculation of $\Theta_k(\theta_i)$ involves highly complicated calculations that will not be provided here.

Profile 4

Consider the linear profile

$$n(r) = a + br, \tag{26}$$

where a and b are constants. This linear refractive index profile has been utilized with respect to absorption measurements of nonlinear optical liquids in the visible and near-infrared spectral region [15]. Surprisingly, the algebraic complexity for this seemingly innocuous profile is even worse than that for profile 3. For this reason we state only the final result for $\Delta(\theta_i)$. In what follows let

$$A = \frac{a}{q}, \; B = \frac{b}{q} \; and \; q = \sin\theta_i, \; \theta_i > 0. \tag{27}$$

We also define

$$k = -\frac{1}{4B}\left[A^2 \pm i\sqrt{\left(A^2 + 4B\right)\left(4B - A^2\right)}\right]. \tag{28}$$

The positive root is taken if $4B - A^2 > 0$ and the negative root if $4B - A^2 < 0$.

Let

$$\sin\phi = \left\{\left[-\frac{\left(A + \sqrt{A^2 + 4B}\right)}{k\left(A - \sqrt{A^2 + 4B}\right)}\right]\left[\frac{-\frac{1}{2}\left(A - \sqrt{A^2 + 4B}\right)r + 1}{\frac{1}{2}\left(A + \sqrt{A^2 + 4B}\right)r - 1}\right]\right\}^{1/2} \tag{29}$$

and

$$\alpha^2 = \frac{k\left(A - \sqrt{A^2 + 4B}\right)}{A + \sqrt{A^2 + 4B}}. \tag{30}$$

The final result is

$$\Delta(\theta_i) = -\frac{\sqrt{2Bk}}{B}\left[\left(\frac{A + \sqrt{A^2 + 4B}}{2}\right)F(\phi, k) - \sqrt{A^2 + 4B}\,\Pi(\phi, \alpha^2, k)\right], \tag{31}$$

where as in equation (22), $\Pi(\phi, \alpha^2, k)$ is the incomplete elliptic integral of the third kind, and

$$F(\phi, k) = \int_0^{\sin\phi} \frac{dt}{\sqrt{\left(1 - t^2\right)\left(1 - k^2 t^2\right)}} \tag{32}$$

is the incomplete elliptic integral of the first kind. From equation (31) we can compute the deviation angle $\Theta_k(\theta_i)$ given by equation (7). Again, though, the analytic expression for $\Theta_k(\theta_i)$ is extremely complicated and will not be presented here.

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Profile 5

Consider next the refractive index profile

$$n(r) = n_1 r^{1/b-1}\left(2 - r^{2/b}\right)^{1/2}, \quad n_1 = n(1) > 1 \cdot \tag{33}$$

This profile is singular at r=0 if one of the following conditions is satisfied:

$b < 0$; $b > 1$.

In order for a kth-order bow to exist, $b > 2$. Consequently, the second singularity condition $b > 1$ will be met in order to guarantee the existence of a zero-order bow. When $b = 1$, this profile corresponds to a Luneburg lens [14]. Defining

$$a \equiv \frac{\sin\theta_i}{n_1}, \tag{34}$$

it follows that

$$r_c^{2/b} = \frac{2 \pm \sqrt{4 - 4a^2}}{2} = 1 \pm \sqrt{1 - a^2}. \tag{35}$$

Because $0 < r_c < r < 1$ the negative root must prevail, since $a^2 < 1$ so that $r_c \in R$. Hence

$$r_c(\theta_i) = \left(1 - \sqrt{1 - a^2}\right)^{b/2}. \tag{36}$$

Ultimately, the relatively simple result is

$$\Delta(\theta_i) = \frac{b}{2}(\pi - \arcsin a), \tag{37}$$

and hence

$$\Theta(\theta_i) = 2\theta_i + \pi\left[(k+1)b - 1\right] - (k+1)b \arcsin\left(\frac{\sin\theta_i}{n_1}\right). \tag{38}$$

As above, for a kth-order bow to exist for some critical angle of incidence θ_k, it is necessary and sufficient that

$$\Theta_k^i\left(\theta_{i_c}\right) = 0 \cdot$$

Thus the condition for extrema is that

$$\cos\theta_{i_c} = 2\left[\frac{\left(n_1^2 - 1\right)}{(k+1)^2 b^2 - 4}\right]^{1/2}. \tag{39}$$

The consequences of this equation for k-th order bows can be readily inferred. In particular, for a zero-order bow (k=0), if we restrict ourselves to the least potentially singular case b>0, then a zero-order bow can exist for this profile if $b \geq 2n_1$. Note that a zero-order bow cannot exist if n_1=1.

Profile 6

Consider the 'reciprocal linear' refractive index profile given by

$$n(r) = (ar + b)^{-1}, \tag{40}$$

where again a and b are constants, and previously considered by Gould and Burman [9] and Adam and Laven [18]. It has applications in atmospheric and terrestrial physics [9]. The profile is singular when $r = -b/a$. It is readily shown that

$$r_c(\theta_i) = \frac{\beta}{1 - \alpha}, \quad \text{where } \alpha = a\sin\theta_i, \quad \beta = \frac{b}{a}\alpha \cdot \tag{41}$$

One resulting form of the basis angle $\Delta(\theta_i)$ is:

$$\Delta(\theta_i) = \frac{\alpha}{\sqrt{1-\alpha^2}} \ln\left[\frac{\sqrt{1-\alpha^2}\sqrt{1-(\alpha+\beta)^2} + 1 - \alpha(\alpha+\beta)}{\beta}\right] + \frac{\pi}{2} - \arcsin(\alpha+\beta). \tag{42}$$

Using the formula

$$\arccos z = -i\log\left(z + i\sqrt{1 - z^2}\right) \tag{43}$$

$\Delta(\theta_i)$ may also be expressed as

$$\Delta(\theta_i) = \frac{\alpha}{\sqrt{1-\alpha^2}} \arccos\left[\frac{1 - \alpha(\alpha+\beta)}{\beta}\right] + \arccos(\alpha+\beta). \tag{44}$$

$$= \frac{a\sin\theta_i}{\sqrt{a^2\sin^2\theta_i - 1}} \arccos\left[\frac{1 - a(a+b)a\sin^2\theta_i}{b\sin\theta_i}\right] + \arccos\left[(a+b)\sin\theta_i\right],$$

where we have utilized the definitions for α and β. Once more, the expression for $\Theta_k(\theta_i)$ follows quite naturally. More details of this profile may be found [19].

Profile 7

Next we examine the refractive index profile

$$n(r) = \frac{a}{r\ln(br)}, \tag{45}$$

where a and b again are constants. This profile is singular at r=0 and $r = b^{-1}$ and is undefined if $b \leq 0$. This refractive index profile has applications in radio wave propagation [13]. If

$$m \equiv \frac{a}{\sin\theta_i},$$

then $r_c(\theta_i)$ is given by

$$r_c(\theta_i) = b^{-1}e^{m}. \tag{46}$$

The basis angle is

$$\Delta(\theta_i) = -\sqrt{a^2\csc^2\theta_i - (\ln b)^2}, \tag{47}$$

and therefore, for this refractive index profile, a kth-order bow exists if

$$1 + \frac{(k+1)a^2\csc^2\theta_{i_c}\cot\theta_{i_c}}{\sqrt{a^2\csc^2\theta_{i_c} - (\ln b)^2}} - 0. \tag{48}$$

Since θ_{i_c} is in the first quadrant, and the trigonometric functions are all positive in the first quadrant, equation (48) cannot be satisfied. Thus a kth-order bow cannot exist for this refractive index profile.

Profile 8

Consider the similar refractive index profile

$$n(r) = \frac{a}{r\sqrt{\ln(br)}}, \tag{49}$$

where a and b once more are real constants. This refractive index profile also has applications in radio wave propagation [13]. It is singular at $r = 0$, $r = b^{-1}$ and undefined for $b \leq 0$, as was the case for the previous profile in equation (45). In addition, it is purely imaginary in the domain (0, b^{-1}) Note that now

$$r_c(\theta_i) = b^{-1}e^{m^2}. \tag{50}$$

Consequently, after judious use of inverse trigonometric functions, it transpires that

$$\Delta(\theta_i) = -\left[\sqrt{\ln b}\sqrt{a^2\csc^2\theta_i - \ln b} + \frac{a^2}{2}\csc^2\theta_i \arccos\left(\frac{2\ln b}{a^2}\sin^2\theta_i - 1\right)\right]. \tag{51}$$

Hence

$$\Delta'(\theta_i) = a^2\csc^2\theta_i\cot\theta_i\left[\frac{2\sqrt{\ln b}}{\sqrt{a^2\csc^2\theta_i - \ln b}} + \arccos\left(\frac{2\ln b}{a^2}\sin^2\theta_i - 1\right)\right]. \tag{52}$$

A kth-order bow exists if

$$1+(k+1)a^2\csc^2\theta_{i_c}\cot\theta_{i_c}\left[\frac{2\sqrt{\ln b}}{\sqrt{a^2\csc^2\theta_{i_c}-\ln b}}+\arccos\left(\frac{2\ln b}{a^2}\sin^2\theta_{i_c}-1\right)\right]=0. \quad (53)$$

The second term in equation (53) is positive due to the definition of θ_{i_c}. In order for it to be satisfied, the last term must be negative, i.e. the inverse cosine function must be negative. However, for the set of real numbers, the inverse cosine function is always positive. Therefore, as with the previous profile, a kth-order bow cannot exist for this profile.

Profile 9

Consider the similar refractive index profile

$$n(r)=\left(ar^{-2}+br^{-1}+c\right)^{1/2}, \quad (54)$$

where a, b, and c are real constants. This profile has applications in atmospheric and terrestrial physics as investigated by [9]. Obviously it is singular at $r = 0$. Let $K = \sin\theta_i$. Then

$$r_c(\theta_i)=\frac{-b\pm\sqrt{b^2-4c\left(a-K^2\right)}}{2c}. \quad (55)$$

It must be the case that $0 \le r_c(\theta_i)<1$ and $r_c(\theta_i) \in R$. In order to guarantee that the radial point of closest approach to the center of the sphere, $r_c(\theta_i)$, is a real quantity, it is required that

$$b^2>4c\left(a-K^2\right).$$

There are numerous cases for this profile where we can study the behavior of the constants that would determine whether we choose the positive or negative sign in equation (55). Since we require that $n(r) \in R$, we will not consider the case where all three constants are negative. Regardless of whether $c > 0$ or $c < 0$, the first condition above would result in the following inequality

$$a\le K^2<a+b+c. \quad (56)$$

In other words it is required that a+b+c>1. Since $0 < K^2 \le 1$, it follows that

$$a>a-K^2\ge a-1. \quad (57)$$

But $a - K^2 < 0$ (and from the equation for $\Delta(\theta_i)$ $K^2 \neq a$) this inequality is satisfied only if a <1. Using $a-K^2 \ge a-1$ in the condition $b^2 > 4c(a-K^2)$ implies that

$$b^2>4c(a-1). \quad (58)$$

Without loss of generality, we assume that $a < 0$, $b > 0$, and $c > 0$. As a result, we must take the positive root in equation (55). We note that a and b must be small and large enough, respectively, so that it is real for all values of r in the domain. Thus we obtain the result

$$\Delta(\theta_i)=\frac{K}{\sqrt{K^2-a}}\arccos\left[-\frac{2\left(a-K^2\right)+b}{\sqrt{b^2-4c\left(a-K^2\right)}}\right], \quad (59)$$

and hence it follows that

$$\Theta_k(\theta_i)=2\theta_i-\pi+2(k+1)\frac{K}{\sqrt{K^2-a}}\arccos\left[-\frac{2\left(a-K^2\right)+b}{\sqrt{b^2-4c\left(a-K^2\right)}}\right]. \quad (60)$$

Profile 10

Finally, consider the power-law refractive index profile

$$n(r)=\alpha r^{\eta}, \quad (61)$$

where η is of either sign, $0<b<r\le a$, and $\alpha>0$. This profile is obviously singular at r=0 if $\eta<0$. In order for a zero-order bow to exist, it will be shown that $\eta<0$. This refractive index profile has been used in connection with a general explanation of both the rainbow ray and the glory ray phenomena in meteorological optics. Specifically, it has been invoked to suggest that melting ice crystals may be strong contributors to the so-called 'glory ray'. In addition, this refractive index profile was studied to provide a more general explanation of both the rainbow ray and the glory ray phenomena by analyzing the scattering processes of inhomogeneous particles [17].

By means of a judicious change of variable, this profile can be transformed into the standard one for a *constant* refractive index. Hence if $K = \sin\theta_i$ we find that

$$\Delta(\theta_i)=K\int_{r_c(\theta_i)}^{1}\frac{dr}{r\sqrt{\alpha^2r^{2(\eta+1)}-K^2}}=\int_{r_c(\theta_i)}^{1}\frac{dr}{r\sqrt{Ar^p-1}} \quad (62)$$

where $p = 2(\eta+1)$. Hence

$$r_c(\theta_i)=\left(\frac{K}{\alpha}\right)^{2/p}. \quad (63)$$

In order to evaluate the integral in equation (62), make the change of variables

$$v^2=Ar^p \quad (64)$$

noting that

$$v\left(r_c(\theta_i)\right)=A^{1/2}\left(r_c(\theta_i)\right)^{p/2}=A^{1/2}\frac{K}{\alpha}=1 \quad (65)$$

with $v(1)=\sqrt{A}$. Using these substitutions, the basis deviation angle becomes

$$\Delta(\theta_i)=\frac{2}{p}\int_1^{A^{1/2}}\frac{dv}{v\sqrt{v^2-1}}=\frac{1}{\eta+1}\left[\operatorname{arcsec}\left(\frac{\alpha}{K}r^{\eta+1}\right)\right]_{r_c(\theta_i)}^{1} \quad (66)$$

where $\eta\neq -1$. Therefore, if

$$n(r)\equiv n_b,\quad 0\le r\le b; \quad (67)$$

$$=\alpha r^n,\quad b\le r\le a, \quad (68)$$

by continuity at $r = b$,

$$\alpha\equiv n_b b^{-\eta} \quad (69)$$

and so

$$\Delta(\theta_i)=\frac{1}{\eta+1}\operatorname{arcsec}\left(\frac{n_b}{Kb^{\eta}}\right). \quad (70)$$

Hence

$$\Theta_k(\theta_i)=2\theta_i-\pi+\frac{2(k+1)}{\eta+1}\operatorname{arcsec}\left(\frac{n_b}{Kb^{\eta}}\right). \quad (71)$$

Since

$$\Delta'(\theta_i)=\frac{-1}{\eta+1}\frac{\cos\theta_i}{\sqrt{\alpha^2-K^2}}$$

a kth-order bow exists if $\Delta'(\theta_i) = -(k+1)^{-1}$. Hence, for the refractive index profile given by the profile (61), a kth-order bow exists if

$$\sin\theta_{i_c}=\sqrt{\frac{(k+1)^2-(\eta+1)^2\alpha^2}{(k+1)^2-(\eta+1)^2}} \quad (72)$$

where $\eta \neq 1$. The implications of this equation for zero-order bows ($k = 0$) in particular are discussed in [19].

Conclusion

A ray-geometric study of the optical properties of radially inhomogeneous media involves evaluation of a certain improper integral; the ray-path integral. This is an improper integral with relatively few known exact analytical forms, even for simple refractive index profiles $n(r)$. In this paper we have evaluated the integral for ten different $n(r)$ profiles. Many of these have not been provided elsewhere (further mathematical details are discussed in [19]). These results are of considerable interest in a variety of optics-related fields. In the accompanying appendix we derive the mirage theorem for horizontally stratified media. This illustrates a fundamental principle in geometrical optics, one which undergirds the evaluation of the ray-path integral, namely that the path of a ray is always concave towards regions of higher refractive index. This also explains, in general terms, the reasons for both inferior and superior mirages observed in nature.

Appendix: The Mirage Theorem

Fermat's principle of least time

For simplicity we will content ourselves with proving the theorem for a horizontally stratified medium [20] because of the obvious connection with mirages, but it can be generalized (but the name mirage theorem is taken from [20]). Suppose that the speed of light – or sound, for that matter; acoustic mirages are also possible, though less easily recognized – depends on position in the medium in which the continuous refractive index varies vertically, i.e. $n=n(y)$, and so can be characterized by the position along the ray trajectory parametrized by arc length s. Let the propagation time from say, (0,0) to x_1,0 be T. Then since

$$\frac{ds}{dt} = c(s) \tag{73}$$

we write

$$T = \int_0^T dt = \int_0^{s(x_1)} \frac{ds}{c(s)} \tag{74}$$

If the speed varies in a given manner, $c = c(y)$ say, where y is the altitude above the surface of a locally flat earth, then we can write c in terms of the refractive index $n(y)$, such that

$$c(y) = \frac{c_0}{n(y)} \tag{75}$$

where c_0 is (for light) the speed of light in vacuo. Thus the refractive index of a medium, if constant, can be defined as the ratio of the speed of light in vacuo to the speed in the medium. The corresponding integral for the travel time T can be written

$$T = \frac{1}{c_0}\int_0^{x_1} n(y)\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} dx \equiv \int_0^{x_1} f\left\{y(x), y'(x)\right\} dx, \tag{76}$$

the integrand being explicitly independent of x. The resulting path traced out by the ray is a consequence of *Fermat's Principle of Least Time*. From the Euler-Lagrange equation for f

$$\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0 \tag{77}$$

In view of fact that the integrand does not depend explicitly on x, this equation reduces to the following first-order partial differential equation:

$$f - y'\left(\frac{df}{dy'}\right) = K \tag{78}$$

where K is a constant. This equation has the solution

$$\frac{n(y)}{(1+y'^2)^{1/2}} = k \tag{79}$$

A simple diagram shows that

$$n(y)\sin\theta = k \tag{80}$$

where θ is the angle between the vertical direction y and the ray path. Note that $K > 0$ because it is the value of the refractive index when the tangent to the ray path is zero, i.e. it is horizontal. (80) is Snell's law of refraction for a continuously-varying refractive index in a plane-stratified medium, a special case of equation (1). Solving equation (79) as an initial-value problem for rays passing through the point (x_0, y_0) it follows that the ray trajectory satisfies

$$x - x_0 = \pm K \int_{y0}^{y} \frac{d\xi}{\left[n^2(\xi) - K^2\right]^{1/2}} \tag{81}$$

where $K > 0$ without loss of generality. Note the similarity between this integral and that in equation (8). Finally we can prove

The mirage theorem: the ray path is concave towards regions of higher refractive index

Proof: From equation (79) it follows that

$$y'^2 + 1 = \frac{n^2(y)}{K^2} \tag{82}$$

Differentiating this with respect to y it follows that

$$y'(x)\left[y''(x) - \frac{n(y)n'(y)}{K^2}\right] = 0, \tag{83}$$

so for $y' \neq 0$ the sign of y'' is the sign of n'. Thus if $n'>0$ (n increases vertically), $y''>0$ and the curve is concave upward; if $n'<0$ (n decreases vertically), $y''<0$ and the curve is concave downward. These are necessary and sufficient conditions.

Competing Interests

The authors declare that they have no competing interests.

Author Contributions

Both the authors substantially contributed to the study conception and design as well as the acquisition and interpretation of the data and drafting the manuscript.

References

1. Adam JA (2011) Zero-order bows in radially inhomogeneous spheres: direct and inverse problems. Appl Opt 50: F50-F59.
2. Adam JA (2013) Rainbows' in homogeneous and radially inhomogeneous spheres: connections with ray, wave, and potential scattering theory. In: Toni B (Ed) *Advances in interdisciplinary mathematical research: applications to engineering, physical and life sciences*, Springer Proceed Math Statis 37: Springer, USA 57-96.
3. Adam JA (2015) Scattering of Electromagnetic Plane Waves in Radially Inhomogeneous Media: Ray Theory, Exact Solutions and Connections with Potential Scattering Theory. In: Kokhanovsky AA (Ed) Light Scattering Reviews 9, Springer, USA 101–132.
4. Born M, Wolf E (1999) Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light. (7th edition), Cambridge: Cambridge University Press, UK.
5. Alexopoulos, NG (1974) On the refractive properties of media with poles or zeros in the index of refraction. IEEE Trans Ant Prop 22: 242–251.
6. Alexopoulos NG (1972) Scattering from inhomogeneous cylindrically symmetric lenses with a line infinity in the index of refraction. J Optical Soc Ame 62: 1088-1094.
7. Alexopoulos NG (1974) On the refractive properties of media with poles or zeros in the index of refraction. IEEE Trans Antennas Propag 22: 242-251.
8. Alexopoulos NG (1971) Scattering from inhomogeneous lenses with singular points. J Opt Soc Ame 61: 876-878.
9. Gould RN, Burman R (1964) Some electromagnetic wave functions for propagation in stratified media. J Atmospheric Terrest Phys 26: 335-340.

10. Vetrano MR, van Beeck JPAJ, Riethmuller ML (2005) Generalization of the rainbow Airy theory to nonuniform spheres. Optics Lett 30: 658-660.

11. Lock JA, Laven P, Adam JA (2015) Scattering of a Plane Electromagnetic Wave by a Generalized Luneburg Sphere. Part 1: Ray Scattering. J Quant Spectro Rad Trans 162: 154-163.

12. Adams MJ (1981) An Introduction to Optical Waveguides. New York USA.

13. Westcott BS (1968) Electromagnetic wave propagation in spherically stratified isotropic media. Electronic Lett 4: 572-573.

14. Leonhardt U, Philbin T (2010) Geometry and Light: The Science of Invisibility. Mineola NY: Dover Publications 278.

15. Kedenburg S, Vieweg, M, Gissibl, T Giessen H (2012) Linear refractive index and absorption measurements of nonlinear optical liquids in the visible and near-infrared spectral region. J Opt Soc Am 2: 1588-1611.

16. Brockman CL (1974) High Frequency Electromagnetic Wave Backscattering from Radially Inhomogeneous Dielectric Spheres. University of California.

17. Brockman CL, Alexopoulos NG (1977) Geometric optics of inhomogeneous particles: glory ray and the rainbow revisited Appl Opt 16: 166-174.

18. Adam JA, Laven P (2007) Rainbows from inhomogeneous transparent spheres: a ray-theoretic approach. Appl Opt 46: 922-929.

19. Pohrivchak MA (2014) Ray and wave-theoretic approach to electromagnetic scattering from radially inhomogeneous spheres and cylinders. ph.d dissertation, Old Dominion University USA.

20. Adam JA (2017) Rays, Waves and Scattering: Topics in Classical Mathematical Physics. Princeton, NJ: Princeton University Press, USA, in press.