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Bessel Tempered Stable Distributions and Processes

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Abstract

We introduce a flexible parametric family of tempering functions for the stable L´evy density and then use it to generate a new parametric class of infinitely divisible distributions and processes that we call Bessel tempered stable(BTS). This class unifies various types of tempered stable processes known in the literature, such as "KoBoL" ,"CGMY", "MTS" and "RDTS". For the BTS distributions, we present their distributional structure: moments, characteristic functions and cumulants. For the BTS processes, we investigate their path structure: the tail behavior of jumps, the p-th variation of sample paths, and short and long time behavior of the process. We also investigate equivalence of measures within the class of the BTS processes. Finally, as an application, exponential stock models driven by BTS processes are discussed. The merit of our model is that unlike known tempered stable models, it contains an additional parameter which enables us to determine a suitable tempering function which is consistent with the observed market data, so that it might help the practitioner to choose a correct model.

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Introduction

Stable processes are a subclass of pure jump Levy processes which are characterized by power tail property and scaling behavior. Due to these two attractive features, they have been used to describe the statistical properties of complex phenomena in applied fields of probability theory, such as statistical physics and financial mathematics[1,2]. However, their applications to empirical data are limited because the stable distributions have infinite second and higher moments, while empirical distributions often have finite moments [3]. For this reason, various modifications of the stable distribution have been proposed by many authors as valid alternative models in the literature. Such modifications were specified by employing a so-called tempering function which alters the stable Levy density so that the resulting Levy density is decaying much faster. As will be explained below, many authors have employed different types of tempering functions to propose various modifications of stable processes that we call "tempered stable"(TS)process [1,4-8]. The first form of such tempering functions was an exponential function introduced by Koponen in [1].

Koponen [1] first introduced the TS process specified by exponential tempering function under the name of "truncated Levy flight" in physics literature, and later Boyardchenko and Levendorskii [4] extended it to the TS process under the name of "KoboL" process . It is worth noting that the Levy density of the TS process is the same as the α-stable Levy density, except for the additional exponential factor, so that the TS process retains the same qualitative properties of $\alpha \in (0, 2)$ as the α-stable process being tempered, while they have different tail distributions.

Carr, Geman, Madan and Yor [5] introduced the pure jump infinitely divisible process under the name of "CGMY" process to obtain more flexible process than the variance gamma(VG) process in [10]. The process was originally obtained as an extension of the VG Levy density where $\alpha=0$, to the CGMY Levy density by adding an additional parameter "α" $\in (-\infty, 2)$ which permits the path structure of the process in the way: the process is of finite activity if and only if $\alpha < 0$, and of infinite activity/finite variation if $0 < \alpha < 1$, and of infinite activity/infinite variation if $1 < \alpha < 2$. It is worth noting that if $\alpha \leq 0$ then the process has no connection with the TS process, however if α is restricted to $(0, 2)$, then the process becomes the TS process specified by an exponential tempering function, indeed it becomes a special case of the KoBoL process. Due to this point, the CGMY process has been categorized as a TS processes in the literature.

In recent years, such TS processes, called "exponentially TS(ETS)" process, have been considerably studied in financial mathematics to model the dynamics of log stock prices [3,4,5,11,12]. As an extension of ETS process, the multidimensional TS process specified by a completely monotone tempering function was introduced and studied by Rosin´ski [6].

More recently, Kim, Rachev, Chung and Bianchi [7] have proposed a new 4-parametric family of pure jump Levy process under the name of "the modified tempered stable (MTS)" processes, specified by a new parametric tempering function which is not exponential, even not completely monotone. It has been shown in [13] that the class of the MTS processes is totally disjoint with that of the ETS processes, and that the VG processes [10] locate at the common boundary of those two classes. It also has been shown in [7] that the MTS processes share many nice structural and analytic properties with the ETS processes, and have slower decay rate of big jumps than the ETS processes. The rapidly decreasing tempered stable (RDTS) process [8] specified by a squared exponential tempering function has been introduced and studied.

In this paper, we first introduce a very general parametric family of tempering functions for the stable distribution which are defined as the mixture of squared exponential function with inverse gamma distribution having shape and scale parameters on positive real line. The tempering functions are two-parameter functions involving the modified Bessel function of the second kind; hence named as the BTS tempering functions. We next use the tempering functions

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to propose a new flexible class of pure jump Levy processes, called Bessel tempered stable (BTS) processes. It is shown that our BTS processes unify various types of the existing classes of TS processes in the literature, such as the classes of KoBoL,CGMY, MTS and RDTS processes. Accordingly we can compare those TS processes more efficiently under one system. The characteristic function of the BTS distribution is obtained in closed form which is expressed in terms of the hyper-geometric functions, so that it can be used to recover the model option prices through the fast Fourier transform [14] when we calibrate an option pricing model to the observed option prices.

As an application of our BTS processes, we describe stock prices driven by a BTS process under both the market and the risk-neutral measure. In recent years, the exponential Levy models have been used for arbitrage-free option pricing. For arbitrage-free option pricing model, the existence of at least one martingale measure is required. In the literature, there have been two different ways to select a martingale measure under which the discounted asset price process becomes a martingale. One approach is to select the martingale measure such that it is equivalent to the market measure. There are many ways to obtain such martingale measures. Among them, the Esscher transform [15] and the minimal entropy martingale measure [16] which preserve the Levy property of log returns [17]. If the market is incomplete, there is a large set of equivalent martingale measures and so we have the set of arbitrage-free option prices which is given as an interval [18]. Thus this approach does not give values which are consistent with the observed market option prices. This drawback leads us to consider the other approach for an option pricing model. The other approach is to select a martingale measure(so-called an implied martingale measure) as the result of a calibration to the market's option prices, without taking account any historical data, that is, directly calibrate a martingale measure under which the option values are consistent with market option prices. Such an approach is often referred to as an inverse problem by the practitioners. For examples of this approach, refer to Carr, Geman, Madan and Yor [5] and [14], where they estimate parameters such that the model prices are consistent with market option prices, using the method of least squared estimation. Under the assumption that option markets are liquid enough, we will be justified to use this approach to model a more realistic dynamics of the option prices.

Preliminaries

A Borel measure M on $\mathbb{R}$ is called a Levy measure if $M(0) = 0$ and, $\int_{\mathbb{R}} (1 \wedge |x^2|) M(dx) < \infty$, where a ^ b= min{a,b}, a,b ∈ $\mathbb{R}$. By the Levy-Kintchine formula, the infinitely divisible random variable is characterized by its characteristic function

$$\phi(u) = \exp\left(i\gamma u - \frac{1}{2}\sigma^2 u + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux 1_{|x|\le 1}) M(dx) \right) \quad (1)$$

where σ ≥ 0, γ ∈ $\mathbb{R}$ and the *Levy measure* M are uniquely determined. Here, the triple (γ, σ^2, M) is called a *Levy triplet*. A stochastic process $(X_t)_{t\ge 0}$ is called a *Levy process* if (i) it has zero value at the initial time (ii) the process has independent increments and stationary increments, (iii) every sample path of the process is right continuous and has left limit, and (iv) the process is stochastically continuous. If $X = (X_t)_{t\ge 0}$ is a Levy process, then X_1 is an infinitely divisible random variable whose Levy triplet is (γ, σ^2, M). Conversely, if Y is an infinitely divisible random variable with Levy triple (γ, σ^2, M), then there exists uniquely a Levy process $(X_t)_{t\ge 0}$ such that $X_1 = Y$ [19]. Such a process $(X_t)_{t\ge 0}$ will be referred to as the Levy process with *Levy triplet* (γ, σ^2, M).

A Levy process with Levy triplet (γ, σ^2, M) is called a pure jump process if σ = 0. A pure jump process is called α-stable if its Levy measure M is given by

$$\mathrm{M}(dx) = M_{stable}(x)\,dx := \left(\frac{C_+}{x^{1+\alpha}} 1_{x>0}(x) + \frac{C_-}{x^{1+\alpha}} 1_{x<0}(x) \right) dx \quad (2)$$

with parameters $\alpha \in (0, 2)$ and $C_\pm \ge 0$ with $C_+ + C_- > 0$ [20,21]. We note that the stable Levy density $M_{stable}(x)$ decays at a polynomial rate on each of (-∞, 0) and (0, ∞). Thus stable distributions have too heavy tails for applications, expecially in financial modelling, so that we need to employ so-called "tempering functions" that make them have lighter tails.

Let us denote W by the class of all continuous functions w : (0, ∞) → [0, ∞) such that : (i) decreasing, (ii) $w(0+) = 1$ and (iii) $\lim_{x\to\infty} x^n w(x) = 0$ for every n ∈ℕ. For $w \in W$, define the associated function q : ℝ→[0,∞) of the form :

$$q(x) = w(\lambda_+ x) 1_{x>0}(x) + w(\lambda_- |x|) 1_{x<0}(x), \quad (3)$$

where $\lambda_\pm \ge 0$. Here, we easily see that q is such that $q(x)\cdot M_{stable}(dx)$ becomes a Levy measure on $\mathbb{R}$. Such a function q will be called a *w-tempering function* for the stable Levy measure, or simply a *w-tempering function*. The Levy measure associated to this *w-tempering function q*, denoted by $M_{TS}(dx)$, is given as

$$\mathrm{q}(\mathrm{x}).M_{stable}(x)\,dx = \left(\frac{C_+ w(\lambda_+ x)}{x^{1+\alpha}} 1_{x>0}(x) + \frac{C_- w(\lambda_- |x|)}{|x|^{1+\alpha}} 1_{x<0}(x) \right) dx \quad (4)$$

This Levy measure $M_{TS}(dx)$ will be referred to as a *tempered stable* (TS) *Levy measure* specified by a w-tempering function q. Due to conditions (i) - (iii) on w, for each $p > 0$, $|x|^p\, C_\pm w(\lambda_\pm |x|)$ is bounded on $\mathbb{R}$, and hence it follows that for all p > 0 and α ∈ (0, 2),

$$\int_{|x|>1} |x|^p \, M_{TS}(dx) < \infty \quad . \quad (5)$$

A pure jump Levy process with the Levy measure (4) will be called a *tempered stable* (TS) *process* specified by a w-tempering function q. From (5) and a result of [19, p.95], it follows that the associated TS distribution have finite moments of all order. Hence we see that TS distributions associated to the class W constitute a family of infinitely divisible distributions without Gaussian part, which are both skewed and leptokurtic with all moments finite.

We can notice that the function $w \in W$ in (4) provides a basic pattern which determines the shape of w-tempering function describing how the Levy density of the TS process decays like. Thus we can see that the function w(·) determines the key feature of the TS processes. We further note that, when we use such a TS process as a financial model, the difference in parameters λ_+, λ_- serve to capture the asymmetry of decay rates for positive and negative jumps of the stock returns in the market.

Now let us review following five classes of TS processes with the Levy measures $M_{TS}(dx)$ specified by particular w-tempering functions, which have been widely used in stochastic modeling with jumps in the literature.:

(a) The KoBoL Process [9]
It is a TS process with $M_{TS}(dx)$ specified by w-tempering function with $w(x) = e^{-x} \in W$ in (4) and α ∈ (0, 2).

- It was originally introduced as an extension of the Truncated Levy flight [1] where $\lambda_- = \lambda_+$.

- The parameter α ∈ (0,2) controls the power law of the corresponding stable distribution.
- The Damped Power Law [3] is a KoBoL Process.

(b) The CGMY Process [5]
It is a TS process with $M_{TS}(dx)$ specified by w-tempering function with $w(x) = e^{-x} \in W$ in (4), $C_- = C_+$, and $\alpha \in (\infty, 2)$.

- It was originally introduced as an extension of the VG process [10] where $\alpha = 0$.
- If $\alpha \in (0, 2)$, it is a special case of the KoBoL process.
- The parameter $\alpha < 2$ captures the structure of paths in such a way that it has finite activity if $\alpha \in (\infty, 0)$, has infinity activity / finite variation if $\alpha \in (0, 1)$ and has infinity activity / infinity variation if $\alpha \in (1, 2)$.

(c) The MTS Process [7, 13]
It is a TS process with $M_{TS}(dx)$ specified by a parametric w-tempering function with $w \in W$ defined by

$$w(x) := q_\alpha(x) = \frac{2^{\frac{1-\alpha}{2}}}{\Gamma\left(\frac{1+\alpha}{2}\right)} |x|^{\frac{1+\alpha}{2}} K_{\frac{1+\alpha}{2}}(|x|),$$

$C_- = C_+$ and $\alpha \in (-1, 2)$, where $K_p(x)$ is the modified Bessel function of the second kind of order $p > 0$, defined on $(0,\infty)$ [22]).

- If $\alpha \in (0, 2)$, the class of the MTS processes is totally disjoint with that of the CGMY processes [13], and has slower decay rate of jumps than that of the CGMY processes.
- Since $q_o(x) = \sqrt{2/\pi}e^{-x}$, the VG process is a special case of the MTS process with $\alpha = 0$.

(d) The RDTS process [8]
It is a TS process with $M_{TS}(dx)$ specified by w-tempering function with $w(x) = e^{-x^2} \in W$ in (4) and $\alpha \in (-1,2)$.

- If α ∈ (0, 2), the RDTS processes have thinner tails than the ETS processes.
- Unlike other types of TS processes, the RDTS processes have their Laplace transforms on the entire real line.

Selection of Tempering Functions for Stable Levy Measure

In this section we shall select a very general parametric family of tempering functions, called Bessel tempering functions, which unify all the particular w-tempering functions being used in tempered stable models mentioned in the above section.

In [6], Rosinski introduced a general class of multivariate TS distribution specified by completely monotone "tempering function" w, meaning that $(-1)^k \frac{d^k w}{dx^k}(x) > 0$, $0 < x < \infty$, $k = 1, 2 \cdots$. Assume that $w(0+) = 1$. Then by Bernstein's theorem [23, 24], it can be described as the mixture of exponential function with a probability measure μ on $(0,\infty)$:

$$w(x) = \int_0^\infty e^{-tx} d\mu(t), 0 < x < \infty. \qquad (6)$$

Hence w is continuous on $[0,\infty)$ with $w(0+) = 1$. Obviously, $\lim_{x\to\infty} x^n w(x) = 0$ for every $n \in \mathbb{N}$, so that $w \in W$. So we can consider a univariate Rosinski's TS distribution whose Levy measure $M_{TS}(dx)$ is of the form (4) where $w(x)$ is given by (6). We thus easily note that this Rosinski's class is a TS process which is more general than previously known some TS classes, such as the classes of KoBoL and CGMY processes specified by $w(x) = e^{-x}$. However, to generate a more flexible W-class of tempering functions for our purpose, we will reparametrize w by replacing μ by the inverse Gamma probability measures whose probability density is given as

$$g(s;\nu,\beta) = \frac{1}{\beta\Gamma(\nu)}\left(\frac{\beta}{s}\right)^{\nu+1} e^{-\frac{\beta}{s}} \; for \; s > 0,$$

where ν, $\beta > 0$ are shape and scale parameters respectively. Then we obtain a parametric family of completely monotone tempering functions w on $(0,\infty)$ which are given as the Laplace transforms of g $(s; \nu, \beta)$:

$$w(x) := q_{\nu,\beta}(x) = \int_0^\infty e^{-sx} g(s;\nu,\beta) ds = \frac{2^{1-\nu}}{\Gamma(\nu)}(\sqrt{4\beta x})^\nu K_\nu(\sqrt{4\beta x}) \qquad (7)$$

[25], where $K_\nu(x)$ is the modified Bessel function of the second kind of order ν [22]. Next, to make this family enlarged, we replace x with x^2, and β with $\lambda^2/4$ ($\lambda > 0$) in Equation (7). Then we get an enlarged family of tempering functions:

$$\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-sx^2} \frac{4}{\lambda^2}\left(\frac{\lambda^2}{4s}\right)^{\nu+1} e^{\frac{-\lambda^2}{4s}} ds = \frac{2^{1-\nu}}{\Gamma(\nu)}(\lambda x)^\nu K_\nu(\lambda x) \; for \; x > 0. \qquad (8)$$

Indeed, it will be shown below that this parametric family of functions of the form in (8) becomes our desired W-class of tempering functions on $(0, \infty)$.

For $\nu > 0$, put $B_\nu(x) := x^\nu K_\nu(x)$ for $x > 0$, Then we have the following facts:

Remark 1. (a) If $\nu > 0$, $B_\nu(0+) = 2^{\nu-1}\Gamma(\nu)$ [17]

(b) $B_{1/2}(\lambda x) = (\lambda x)^{1/2} K_{1/2}(\lambda x) = \sqrt{\frac{\pi}{2}} e^{-\lambda x}$ for $\lambda > 0$.[14]

(c) We note from [25] that the mean and variance of g (s, ν, β) are, respectively,

$$\frac{\beta}{\nu - 1} \text{ for } \nu > 1, \text{ and } \frac{\beta^2}{(\nu-1)^2(\nu-2)} \text{ for } \nu > 2,$$

so that we easily see that g (s; ν, $a\nu$) → δ (s-a) as $\nu \to \infty$, where δ is the Dirac delta function. Hence using this fact, it follows from Equation (8) that

$$\lim_{\nu\to\infty} \frac{2^{1-\nu}}{\Gamma(\nu)} B_\nu(2\lambda\sqrt{\nu}x) = \lim_{\nu\to\infty} \frac{2^{1-\nu}}{\Gamma(\nu)}(2\lambda\sqrt{\nu}x)^\nu K_\nu(2\lambda\sqrt{\nu}x) = e^{-\lambda^2 x^2}.$$

(d) Using the change of variable $s = 1/(4t)$ in the integral in (8), we deduce the well-known integral representation in [17]:

$$B_\nu(x) = x^\nu K_\nu(x) = 2^{\nu-1}\int_0^\infty e^{-\frac{x^2}{4t}} t^{\nu-1} e^{-t} dt, \nu > 0. \qquad (9)$$

Proposition 2. *Suppose $\nu > 0$. The function $B_\nu(.)$ is completely monotonic over $(0,\infty)$ if and only if $0 < \nu \le \frac{1}{2}$.*

Proof. The proof of "if part" follows from the Bernstein theorem with the following integral representations of $x^\nu K_\nu(x)$ [22, 9. 6. 13]:

$$B_\nu(x) = \begin{cases} x^\nu K_{-\nu}(x) = \frac{2^\nu\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-\nu\right)}\int_1^\infty e^{-xt}(t^2-1)^{-\nu-\frac{1}{2}} dt, & 0<\nu<\frac{1}{2} \\ \sqrt{\frac{\pi}{2}} e^{-x}, & \nu=\frac{1}{2} \end{cases}.$$

To prove "only if " part, we take the contra-positive argument. Suppose that $v>1/2$. Using $\frac{d}{dx}(x^v K_v(x)) = -x^v K_{v-1}(x), x>0$, and the recurrence identity [14, p 148]:

$$K_v(x) = K_{v-2}(x) + \frac{2(v-1)}{x} K_{v-1}(x), x>0,$$

we obtain

$$\frac{d^2}{dx^2}(x^v K_v(x)) = x^v K_v(x) - (2v-1)x^{v-1} K_{v-1}(x), x>0$$

so that we can show that

$$\lim_{x\to 0+} \frac{d^2}{dx^2}(x^v K_v(x)) = \begin{cases} -\infty, & \frac{1}{2}<v\le 1 \\ -2^{v-2}\Gamma(v-1)<0, & v>1 \end{cases}$$

Hence, for every $v > \frac{1}{2}$, there exists an $\epsilon_v > 0$ such that $\frac{d^2}{dx^2}(x^v K_v(x)) < 0$ for all $x \in (0, \epsilon_v)$, which implies that $x^v K_v(x)$ cannot be completely monotone, so that we complete the proof of "if only part".

In the sequel, for $v > 0$, $w_v : [0, \infty) \to [0, \infty)$ denotes the function defined by

$$w_v(x) = \frac{2^{1-v}}{\Gamma(v)} x^v K_v(x).$$

Then by Remark 1, we note that w_v is continuous on $[0,\infty)$ with $w_v(0)=1$, and $w_{1/2}(x)=e^{-x}$. Also by Proposition 2, we note that if $v>1/2$, then w_v is not completely monotone , while $w_v(\sqrt{\cdot})$ is completely monotone. Furthermore, due to the asymptotic behavior of K_v [17]. we see that for any $v > 0$,

$$w_v(x) \sim \frac{2^{1-v}}{\Gamma(v)}\sqrt{\frac{\pi}{2}}\frac{e^{-x}}{x^{(1/2)-v}} \text{ as } x\to\infty, \tag{10}$$

so that w_v satises condition (iii) to be in W. From (10) it follows that as v increases, the w-tempering function w_v decreases more slowly as $x\to\infty$.

Now, for fixed w_v, define the following types of functions on $\mathbb{R}$:

$$q_1(x) := w_v(\lambda x)1_{x>0}(x) + w_v(\lambda|x|)1_{x<0}(x), \tag{11}$$

$$q_2(x) := w_v(\lambda_+ x)1_{x>0}(x) + w_v(\lambda_-|x|)1_{x<0}(x), \tag{12}$$

where $v > 0$, λ , $\lambda_+,\lambda_- \ge 0$. Then due to (10), we see that $w_v \in W$. Thererefore, each function q_i is of the form (3), and hence becomes a (two-sided) w-tempering funcion. Here, such functions will be named as *Bessel tempering functions*, because w_v involves Bessel functions K_v of the second kind of order v.

It is worth noting that by specifying appropriately the shape parameter v of Bessel tempering functions, we can obtain, as special cases, all the particular w-tempering functions mentioned in the second section.

The Class of Bessel Tempered Stable Distributions

In this section, we first build two Levy measures on $\mathbb{R}$ specified by the Bessel tempering functions introduced in the previous section, and then use them to generate a very flexible class of real infinitely divisible distributions, called Bessel tempered stable, and investigate their distributional structure: moments, exponential moments, characteristic functions and cumulants.

We shall begin with building two Borel measures associated to Bessel tempering functions q_1 and q_2 as follows :

First, for fixed $\alpha< 2$, C, $v> 0$ and $\lambda \ge 0$, define the symmetric one, $M^{(s)}_{BTS}(dx)$ as

$$q_1(x)\cdot M^{(s)}_{stable}(x)dx = C\left(\frac{w_v(\lambda x)}{x^{1+\alpha}}1_{x>0}(x) + \frac{w_v(\lambda|x|)}{|x|^{1+\alpha}}1_{x<0}(x)\right)dx \tag{13}$$

Next, for fixed $\alpha< 2$, $C_\pm$, $v> 0$ and $\lambda_\pm \ge 0$, define the asymmetric one, $M_{BTS}(dx)$ as

$$q_2(x)\cdot M_{stable}(x)dx = \left(\frac{C_+ w_v(\lambda_+ x)}{x^{1+\alpha}}1_{x>0}(x) + \frac{C_- w_v(\lambda_-|x|)}{|x|^{1+\alpha}}1_{x<0}(x)\right)dx \tag{14}$$

One can show that due to (10), these two Borel measure indeed become TS Levy measures. These TS Levy measures will be referred to as *Bessel tempered stable* (BTS) *Levy measues*. It is worth noting that due to (10), as v increases, the tails of M_{BTS} decay faster, so that we may say that the parameter v controls the extent to which the tails of the Levy measure are tempered.

Now we are ready to introduce a new class of TS distributions specified by BTS Levy measures.

Denition 3. *Let X be an infinitely divisible random variable with Levy triplet (γ, 0, M) and α < 2. Then (i) X is said to follow the symmetric Bessel tempered stable (SBTS) distribution, denoted by X ~ SBTS(α, C, v, λ,γ) if* $M(dx) = M^{(s)}_{BTS}(dx)$,

(ii) X is said to follow the Bessel tempered stable (BTS) *distribution, denoted by X ~* BTS(*α*, $C_\pm$, *v*, $\lambda_\pm$,*γ*) *if* $M(dx) = M_{BTS}(dx)$.

We note that the BTS distribution is asymmetric if either $C_-\neq C_+$ or $\lambda_+\neq\lambda_-$ and is skewed to the right (resp. left) if $\lambda_+< \lambda_-$ (resp. $\lambda_+>\lambda_-$). Thus we see that the difference in λ_+ *and* λ_- determines the asymmetry of the BTS distribution, and hence the BTS distribution is an asymmetric form of the symmetric BTS distribution. Another form of asymmetric BTS distribution will be investigated in a subsequent paper.

Example 4. *Many previously known TS distributions are special cases of our BTS distributions:*

(a) For $v > 0$ *and* $\lambda_\pm = 0$, we get the stable distribution.

(b) For $v = 1/2$, $\alpha = 0$ *and* $C_- = 0$, we get the Gamma distribution. For $v = 1/2$, $\alpha = 1/2$, and $C_- = 0$, we get the inverse Gaussian distribution.

(c) For $v = 1/2$, $\alpha = 0$, and $C_+ = C_-$, we get the Variance Gamma *(VG)* distribution [26]. For $\alpha = 1/2$ and $\alpha = 0$, we get the Bilateral Gamma distribution [12,36].

(d) For $v = 1/2$ and $\alpha \in (0,2)$, we get the KoBol distribution [9]. For $v = 1/2$, $\alpha < 2$ *and* $C_+ = C_-$, we get the CGMY distribution [5].

(e) For $v = 1/2$ *and* $\alpha = -1$, we get the Kou distribution [27].

(f) *If* $0 <v \le 1/2$ *and* $C_+ = C_-$, then w_v is completely monotone by Proposition 2, and hence such BTS distributions are included in the Rosinski's class of univariate tempered stable distributions in [6].

(g) For $v = (\alpha + 1)/2$ and $-1 < \alpha < 2$, we get the MTS distribution.

(h) If $C_+ = C_-$, then by replacing λ_+ and λ_- with $2\lambda_+\sqrt{v}$ and $2\lambda_-\sqrt{v}$, respectively and using by Remark 1, (c), we have

$$\lim_{v\to\infty} M_{BTS}(dx) = C\left(\frac{e^{-\lambda_+^2 x^2}}{x^{1+\alpha}}1_{x>0}(x) + \frac{e^{-\lambda_-^2 x^2}}{|x|^{1+\alpha}}1_{x<0}(x)\right)dx.$$

so that the RDTS distributions are included in the class of BTS distributions.

(i) By (8), the BTS distributions are clearly included in the class of TID distributions [28].

Lemma 5. For $\alpha < 2$ and $p > 0$, the BTS *Levy measures satisfy the following properties*:

(a) if $\alpha < p$, $\int_{-\infty}^{\infty} |x|^p M_{BTS}(dx) < \infty$. In particular, if $p = 2$, then we have

$$\int_{-\infty}^{\infty} |x|^2 M_{BTS}(dx) = \left(\frac{C_+}{\lambda_+^{2-\alpha}} + \frac{C_-}{\lambda_-^{2-\alpha}}\right)\frac{2^{1-\alpha}}{\Gamma(v)}\Gamma\left(v + \frac{2-\alpha}{2}\right)\Gamma\left(\frac{2-\alpha}{2}\right)$$

which gives the quadratic variation contributed by all the jumps of BTS distribution.

(b) if $\alpha < 0$, $M_{BTS}(\mathbb{R}) < \infty$, while if $\alpha > 0$, $M_{BTS}(\mathbb{R}) = \infty$.

Proof. The proof of (a) can be done by using the known integral formular [22, 11.4.22]

$$\int_0^{\infty} x^{\mu} K_v(x)dx = 2^{\mu-1}\Gamma\left(\frac{1+\mu+v}{2}\right)\Gamma\left(\frac{1+\mu-v}{2}\right), \text{ when } 1 + \mu \pm v > 0. \tag{15}$$

If $\alpha < p$, then we have

$$\int_0^{\infty} x^p \left(\frac{w_v(\lambda_\pm x)}{x^{1+\alpha}}\right)dx = \left(\frac{2^{p-\alpha-1}}{\lambda_\pm^{p-\alpha}\Gamma(v)}\right)\Gamma\left(v + \frac{p-\alpha}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right) < \infty$$

which in turn implies

$$\int_{-\infty}^{\infty} |x|^p M_{BTS}(dx) = C_+\int_0^{\infty}\frac{w_v(\lambda_+ x)}{x^{1+\alpha-p}}dx + C_-\int_0^{\infty}\frac{w_v(\lambda_- x)}{x^{1+\alpha-p}}dx < \infty.$$

This completes the proof.

The proof of (b) is an immediate consequences of (a) with $p = 0$.

Proposition 6. (Moments) *Let* $X \sim \text{BTS}(\alpha, C_\pm, v, \lambda_\pm, \gamma)$ and $\alpha < 2$, Then it holds that

(a) $E[|X|^p] < \infty$ for any $p > 0$.

(b) $E[e^{\beta|X|}] < \infty$ for any $\beta \in \mathbb{R}$ with $|\beta| < \lambda_+ \wedge \lambda_-$.

Proof. (a) By the result of [19, Theorem 25.3], it suffices to show that $\int_1^{\infty} x^p M_{BTS}(dx) < \infty$. But it can be shown by using the facts:

$$x^p \frac{C_+ w_v(\lambda_+ x)}{x^{1+\alpha}}dx \sim C_+\frac{2^{1-v}}{\Gamma(v)}\sqrt{\frac{\pi}{2}}x^p\frac{e^{-\lambda_+ x}}{x^{(3/2)+\alpha-v}} \text{ as } x \to \infty,$$

for any $\alpha < 2$ and $v > 0$, and $\int_0^{\infty} x^N e^{-x}dx = \Gamma(N+1)$ for any positive integer N.

(b) As in the proof of (a), it suffces to show that for any $0 < \beta < \lambda_+$, $\int_1^{\infty} e^{\beta x}\frac{(\lambda_+ x)^v K_v(\lambda_+ x)}{x^{1+\alpha}}dx < \infty$. But it can be shown by using the facts: $e^{\beta x}M_{BTS} \sim C_+\frac{2^{1-v}}{\Gamma(v)}\sqrt{\frac{\pi}{2}}\frac{e^{-(\lambda_+-\beta)x}}{x^{(3/2)+\alpha-v}}$ $x \to \infty$ for any $\alpha < 2$ and $v > 0$, and $\int_0^{\infty} x^N e^{-x}dx = \Gamma(N+1)$ for any positive integer N.

Remark 7. (a) We see that $m := \gamma + \int_{|x|>1} x\, M_{BTS}(dx)$ is well-defined by Lemma 5. Hence by Levy-Kintchine formula, we may rewrite the characteristic function of X with Levy triplet $(\gamma, 0, M_{BTS})$ as

$$\phi_X(u) = E\left[e^{iuX}\right] = \exp\left(ium + \int_{-\infty}^{\infty}\left(e^{iux} - 1 - iux\right)M_{BTS}(dx)\right) \tag{16}$$

(b) We note here, from a result of [19, p.95] that for $\alpha \in [0, 2)$, the SBTS and BTS distributions are selfdecomposable. This extends the well-known fact that every stable distribution is selfdecomposable.

The characteristic function of the BTS distribution is obtained in closed form.

Theorem 8. Let $X \sim \text{BTS}(\alpha, C_\pm, v, \lambda_\pm, \gamma)$ with $\alpha \in (-\infty, 2)\setminus\{0,1\}$ and $\lambda_\pm > 0$. Then the *characteristic function* ϕ_X of X is given as

$$\phi_X(u) = E\left[e^{iuX}\right] = \exp\left(ium + C_+ I(iu; \alpha, v, \lambda_+) + C_- I(-iu; \alpha, v, \lambda_-)\right)$$

where $m := \gamma + \int_{|x|>1} x\, M_{BTS}(dx)$,

$$I(s;\alpha,v,\lambda) := \frac{\lambda^{\alpha}}{2^{\alpha+1}\Gamma(v)}\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(v-\frac{\alpha}{2}\right)\left(F\left(v-\frac{\alpha}{2},-\frac{\alpha}{2};\frac{1}{2};\frac{s^2}{\lambda^2}\right)-1\right)$$
$$+\frac{\lambda^{\alpha-1}s}{2^{\alpha}\Gamma(v)}\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(v+\frac{1-\alpha}{2}\right)\left(F\left(v+\frac{1-\alpha}{2},\frac{1-\alpha}{2};\frac{3}{2};\frac{s^2}{\lambda^2}\right)-1\right)$$

and $F := {}_2F_1$ *denotes for hyper-geometric function* [22].

Proof. We first note that

$$\int_{-\infty}^{\infty}\left(e^{iux} - 1 - iux\right)M_{BTS}(dx)$$
$$= \int_0^{\infty}\left(e^{iux} - 1 - iux\right)M_{BTS}(dx) + \int_{-\infty}^{0}\left(e^{iux} - 1 - iux\right)M_{BTS}(dx)$$
$$= C_+ I(iu;\alpha,v,\lambda_+) + C_- I(-iu;\alpha,v,\lambda_-)$$

where $I(s;\alpha,v,\lambda) := \int_0^{\infty}\left(e^{sx} - 1 - sx\right)\frac{w_v(\lambda x)}{x^{1+\alpha}}dx$. Now let's compute this integral:

$$I(s;\alpha,v,\lambda) = \frac{2^{1-v}}{\Gamma(v)}\int_0^{\infty}\left(e^{sx} - 1 - sx\right)\frac{(\lambda x)^v K_v(\lambda x)}{x^{1+\alpha}}dx$$
$$= \frac{2^{1-v}}{\Gamma(v)}\int_0^{\infty}\sum_{n=2}^{\infty}\left(\frac{(sx)^n}{n!}\right)\frac{(\lambda x)^v K_v(\lambda x)}{x^{1+\alpha}}dx$$

By the change of variable $y = \lambda x$, we have

$$I(s;\alpha,v,\lambda) = \frac{2^{1-v}}{\Gamma(v)}\int_0^{\infty}\sum_{n=2}^{\infty}\left(\frac{(sy/\lambda)^n}{n!}\right)\frac{y^v K_v(y)}{\lambda(y/\lambda)^{1+\alpha}}dy$$
$$= \lambda^{\alpha}\frac{2^{1-v}}{\Gamma(v)}\sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{s}{\lambda}\right)^n\int_0^{\infty} y^{n+v-\alpha-1}K_v(y)dy.$$

By (15), we obtain

$$I(s;\alpha,v,\lambda) = \lambda^{\alpha}\frac{2^{1-v}}{\Gamma(v)}\sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{s}{\lambda}\right)^n 2^{x+v-\alpha-2}\Gamma\left(\frac{n-\alpha+2v}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)$$
$$= \frac{\lambda^{\alpha}}{2^{\alpha+1}\Gamma(v)}\sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{2s}{\lambda}\right)^n\Gamma\left(v+\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right) \tag{17}$$

Decomposing the summation into the sums of even and odd terms, we have

$$I(s;\alpha,v,\lambda) = \frac{\lambda^{\alpha}}{2^{\alpha+1}\Gamma(v)}\sum_{n=1}^{\infty}\frac{1}{(2n)!}\left(\frac{2s}{\lambda}\right)^{2n}\Gamma\left(v+n-\frac{\alpha}{2}\right)\Gamma\left(n-\frac{\alpha}{2}\right)$$
$$+\frac{\lambda^{\alpha}}{2^{\alpha+1}\Gamma(v)}\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}\left(\frac{2s}{\lambda}\right)^{2n+1}\Gamma\left(v+n+\frac{1-\alpha}{2}\right)\Gamma\left(n+\frac{1-\alpha}{2}\right).$$

Let $(.)_n$ denote the symbol defined as $(a)_n := \frac{\Gamma(n+a)}{\Gamma(a)}$ for $a \in \mathbb{R}$. Then using this symbol, we have

$$I(s;\alpha,v,\lambda) = \frac{\lambda^{\alpha}}{2^{\alpha+1}\Gamma(v)}\left[\begin{array}{l}\Gamma\left(v-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right)\sum_{n=1}^{\infty}\frac{1}{(2n)!}\left(\frac{2s}{\lambda}\right)^{2n}\left(v-\frac{\alpha}{2}\right)_n\left(-\frac{\alpha}{2}\right)_n + \\ \Gamma\left(v+\frac{1-\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}\left(\frac{2s}{\lambda}\right)^{2n+1}\left(v+\frac{1-\alpha}{2}\right)_n\left(\frac{1-\alpha}{2}\right)_n\end{array}\right]$$

From the identities : $(2n)! = 2^{2n}n!\left(\frac{1}{2}\right)_n$ *and* $(2n+1)! = 2^{2n}n!\left(\frac{3}{2}\right)_n$, this is rewritten as

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$$I(s;\alpha,v,\lambda)=\frac{\lambda^{\alpha}}{2^{\alpha+1}\Gamma(v)}\Gamma\left(v-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{s^2}{\lambda^2}\right)^n\frac{\left(v-\frac{\alpha}{2}\right)_n\left(-\frac{\alpha}{2}\right)_n}{\left(\frac{1}{2}\right)_n}$$

$$+\frac{\lambda^{\alpha}}{2^{\alpha+1}\Gamma(v)}\frac{2s}{\lambda}\Gamma\left(v+\frac{1-\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{s^2}{\lambda^2}\right)^n\frac{\left(v+\frac{1-\alpha}{2}\right)_n\left(\frac{1-\alpha}{2}\right)_n}{\left(\frac{3}{2}\right)_n}$$

$$=\frac{\lambda^{\alpha}}{2^{\alpha+1}\Gamma(v)}\Gamma\left(v-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{s^2}{\lambda^2}\right)^n\frac{\left(v-\frac{\alpha}{2}\right)_n\left(-\frac{\alpha}{2}\right)_n}{\left(\frac{1}{2}\right)_n}-1\right)$$

$$+\frac{\lambda^{\alpha-1}s}{2^{\alpha}\Gamma(v)}\Gamma\left(v+\frac{1-\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{s^2}{\lambda^2}\right)^n\frac{\left(v+\frac{1-\alpha}{2}\right)_n\left(\frac{1-\alpha}{2}\right)_n}{\left(\frac{3}{2}\right)_n}-1\right)$$

From the denition of the hypergeometric function in [22], we have

$$I(s;\alpha,v,\lambda)=\frac{\lambda^{\alpha}}{2^{\alpha+1}\Gamma(v)}\Gamma\left(v-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right)\left(F\left(v-\frac{\alpha}{2},-\frac{\alpha}{2};\frac{1}{2};\frac{s^2}{\lambda^2}\right)-1\right)$$

$$+\frac{\lambda^{\alpha-1}s}{2^{\alpha}\Gamma(v)}\Gamma\left(v+\frac{1-\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\left(F\left(v+\frac{1-\alpha}{2},\frac{1-\alpha}{2};\frac{3}{2};\frac{s^2}{\lambda^2}\right)-1\right)$$

which completes the proof.

We remark in the above theorem that ϕ_X takes a closed form for the special case of $\alpha = 0$, $v = 1/2$ [7, Theorem 2.1]), and so does for the other special case of $\alpha = 1$ when X is symmetric.

The following result shows that the SBTS distribution converges in distribution to the RDTSdistribution as $v\to\infty$.

Corollary 9. Let $\alpha \in (0, 2)\setminus\{1\}$. For $C_+ = C_- = C$, *the distribution of SBTS(* α, v, C, $\lambda_\pm$, γ *) weakly converges to the distribution of RDTS (* α, C, λ_+, λ_-, γ *) as* $v\to\infty$.

Proof. Let $x > 0$, We observe that for $\lambda > 0$, $w\ (2\ \sqrt{vx}) \to e$ as $v\to\infty$, and that the function $I(s;\alpha,v,\lambda)$ in Theorem 8 with replacing λ by $2\lambda\sqrt{v}$ becomes

$$I\left(s;\alpha,v,2\lambda\sqrt{v}\right)=\frac{1}{2}\lambda^{\alpha}\frac{\Gamma\left(v-\frac{\alpha}{2}\right)}{\Gamma(v)v^{-\frac{\alpha}{2}}}\Gamma\left(-\frac{\alpha}{2}\right)\left(F\left(v-\frac{\alpha}{2},-\frac{\alpha}{2};\frac{1}{2};\frac{s^2}{4\lambda^2 v}\right)-1\right)$$

$$+\frac{1}{2}\lambda^{\alpha-1}s\frac{\Gamma\left(v+\frac{1-\alpha}{2}\right)}{\Gamma(v)v^{\frac{1-\alpha}{2}}}\Gamma\left(\frac{1-\alpha}{2}\right)\left(F\left(v+\frac{1-\alpha}{2},\frac{1-\alpha}{2};\frac{3}{2};\frac{s^2}{4\lambda^2 v}\right)-1\right).$$

Now by using the known facts [29, 3.5.2]):

(i) $\lim_{v\to\infty}\frac{\Gamma(v+r)}{\Gamma(v)v^r}=1$, for any real number r.

(ii) $\lim_{a\to\infty}{}_2F_1\left(a,b,c;\frac{x}{a}\right)=M(b,c;x)$, where M is the confluent hyper geometric function, we obtain that $\lim_{v\to\infty} I(s,\alpha,v,2\lambda\sqrt{v}) =$ $G(s,\alpha,\lambda)$, where

$$G(s;\alpha,\lambda)=\frac{1}{2}\lambda^{\alpha}\Gamma\left(-\frac{\alpha}{2}\right)\left(M\left(-\frac{\alpha}{2};\frac{1}{2},\frac{s^2}{4\lambda^2}\right)-1\right)$$

$$+\frac{1}{2}\lambda^{\alpha}s\Gamma\left(\frac{1-\alpha}{2}\right)\left(M\left(\frac{1-\alpha}{2};\frac{3}{2},\frac{s^2}{4\lambda^2}\right)-1\right).$$

Thus the characteristic function of the distribution of *RDTS* (α, C, λ_+, λ_-, γ) is given as

$$\exp\left[ium+CG(iu;\alpha,\lambda_+)+CG(-iu;\alpha,\lambda_-)\right],$$

(see [8]) so that we complete the proof.

The kth-*cumulant* of a random variable X is defined by

$$c_k(X)=\frac{1}{i^k}\frac{d^k}{du^k}\log\phi_X(u)\Big|_{u=0}.$$

The following proposition gives the formula for the k-th cumulants of the BTS distibutions.

Proposition 10. Let $X \sim$ BTS (α, $C_\pm$, v, $\lambda_\pm$, γ) *with* $\alpha \in (-\infty, 2)\setminus\{0,1\}$. *Then the cumulant* $c_k(X)$ *of* X *is given as*

$$c_k(X)=\begin{cases} m:=\gamma+\int_{|x|>1}xM_{BTS}dx, & k=1\\ \frac{2^{k-\alpha-1}}{\Gamma(v)}\Gamma\left(v+\frac{k-\alpha}{2}\right)\Gamma\left(\frac{k-\alpha}{2}\right)\left(C_+\lambda_+^{\alpha-k}+(-1)^k C_-\lambda_-^{\alpha-k}\right), & k=2,3,\ldots\end{cases}$$

Proof. By (17), the characteristic exponent $\log\phi_X(u)$ of X is given by

$$\log\phi_X(u)$$

$$=ium+\frac{1}{2^{\alpha+1}\Gamma(v)}\sum_{n=2}^{\infty}\frac{1}{n!}\left(C_+\lambda_+^{\alpha}\left(\frac{2iu}{\lambda_+}\right)^n+C_-\lambda_-^{\alpha}\left(\frac{-2iu}{\lambda_-}\right)^n\right)$$

$$\cdot\Gamma\left(v+\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right).$$

Clearly, $c_1(X) = $ m. If $n \geq 2$, we have

$$\frac{d^k}{du^k}\log\phi_X(u)$$

$$=\frac{1}{2^{\alpha+1}\Gamma(v)}\sum_{n=k}^{\infty}\frac{n(n-1)\ldots(n-k+1)}{n!}\Gamma\left(v+\frac{n-\alpha}{2}\Gamma\left(\frac{n-\alpha}{2}\right)\right)$$

$$\times\left(C_+\lambda_+^{\alpha}\left(\frac{2iu}{\lambda_+}\right)^{n-k}\left(\frac{2i}{\lambda_+}\right)^k+C_-\lambda_-^{\alpha}\left(\frac{-2iu}{\lambda_-}\right)^{n-k}\left(-\frac{2i}{\lambda_-}\right)^k\right).$$

Therefore, we obtain

$$\frac{1}{i^k}\frac{d^k}{du^k}\log\phi_X(u)\Big|_{u=0}=\frac{2^{k-\alpha-1}}{\Gamma(v)}\Gamma\left(v+\frac{k-\alpha}{2}\right)$$

$$\cdot\Gamma\left(\frac{k-\alpha}{2}\right)\left(C_+\lambda_+^{k-\alpha}+(-1)^k C_-\lambda_-^{\alpha-k}\right),$$

for $k = 2, 3\ldots$, so that we completes the proof.

From the above formula, we can specify the first four cumulants of X:

mean of $X = c_1(X) = m$

$$\text{variance of } X=c_2(X)=\frac{2^{1-\alpha}}{\Gamma(v)}\Gamma\left(v+1-\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)\left(C_+\lambda_+^{\alpha-2}+C_-\lambda_-^{\alpha-2}\right)$$

$$\text{skewness of } X=\frac{c_3(X)}{(c_2(X))^{3/2}}=\frac{2^{\frac{1+\alpha}{2}}\sqrt{\Gamma(v)}\Gamma\left(v+\frac{3-\alpha}{2}\right)\Gamma\left(\frac{3-\alpha}{2}\right)\left(C_+\lambda_+^{\alpha-3}-C_-\lambda_-^{\alpha-3}\right)}{\left(\Gamma\left(v+1-\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)\left(C_+\lambda_+^{\alpha-2}+C_-\lambda_-^{\alpha-2}\right)\right)^{3/2}}$$

$$\text{kurtosis of } X=\frac{c_4(X)}{(c_2(X))^2}=\frac{2^{1+\alpha}\Gamma(v)\Gamma\left(v+2-\frac{\alpha}{2}\right)\Gamma\left(2-\frac{\alpha}{2}\right)\left(C_+\lambda_+^{\alpha-4}+C_-\lambda_-^{\alpha-4}\right)}{\left(\Gamma\left(v+1-\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)\left(C_+\lambda_+^{\alpha-2}+C_-\lambda_-^{\alpha-2}\right)\right)^2}$$

Bessel Tempered Stable Process and their Path Properties

In this section we introduce a new flexible class of pure jump Levy processes associated to the Bessel tempered stable distributions, and then investigate their path structure : the decaying behavior of big size jumps, the p-th variation of sample paths, and short and long time behavior.

Let $\alpha < 2$. A Levy process $(X_t)_{t\geq 0}$ is called the *Bessel tempered stable* (BTS) process with parameters (α, $C_\pm$, v, $\lambda_\pm$, γ) if X_1 follows the infinitely divisible distribution with Levy triplet (γ, 0, M_{BTS}) such that $X_1 \sim$ BTS (α, $C_\pm$, v, $\lambda_\pm$, γ). For a BTS process $(X_t)_{t\geq 0}$, we shall also write $(X_t)_{t\geq 0} \sim$ BTS (α, $C_\pm$, v, $\lambda_\pm$, γ).

Example 11.

(a) For $\alpha \in (0, 2)$, the BTS process with parameters (α, $C_\pm$, $1/2$, $\lambda_\pm$, γ) with $\lambda_\pm = 0$ becomes an α-stable process.

(b) The BTS process with parameters (-1, $C_\pm$, 1/2, $\lambda_\pm$, γ) is a Kou process [27].

(c) The BTS processes with parameters (0, $C_\pm$, 1/2, $\lambda_\pm$, 0) and (0, C, 1/2, $\lambda_\pm$, γ) are, respectively, a bilateral Gamma process [12] and a VG process [10,26].

(d) For $\alpha \in (0, 2)$, the BTS process with parameters (α, $C_\pm$, 1/2, $\lambda_\pm$, γ) is a KoBoL process [9].

(e) For $\alpha \in (-\infty, 2)$, the BTS process with parameters (α, C, 1/2, $\lambda_\pm$, γ) is a CGMY process [5].

(f) The BTS process with parameters (0, C, 1/2, $\lambda_\pm$, γ) weakly converges to the Gamma process as $\lambda_- \to\infty$. The BTS process with parameters (3/2, C, 1/2, $\lambda_\pm$, γ) weakly converges to the inverse Gaussian process as $\lambda_- \to\infty$.

(g) For $\alpha \in (-1, 2)$, the BTS process with parameters (α, C, $(1+\alpha)/2$, $\lambda_\pm$, γ) is a MTS process [7,13] .

(h) For $\alpha \in (0, 2)\backslash\{1\}$. the BTS process with parameters (α, C, v, $2\lambda_+\sqrt{v}$, $2\lambda_-\sqrt{v}$, γ) weakly converge to the RDTS process as $v\to\infty$ [8].

The following result gives some information about how jumps of the BTS processes behave like.

Proposition 12. Let $(X_t)_{t\geq 0} \sim$ BTS (α, $C_\pm$, v, $\lambda_\pm$, γ) *with* $\alpha < 2$. *Then, for fixed* $\mathbb{C}_\pm$, $\lambda_\pm$ and $v > 0$, it holds that

(a) $\alpha \in (0, 2)$, $M_{BTS}(x) \sim M_{stable}(x)$, *as* $|x|\to 0$, *This means that small jumps of the BTS process behave like the α-stable process.*

(b) $\alpha \in (0, 2)$, $M^{\pm}_{BTS}(x) \sim (|x|)^{v-1/2} L(|x|) M^{\pm}_{KoBol}(x)$, as $|x|\to\infty$, where $M^{\pm}_{BTS}(x) = C_\pm \dfrac{w_v(\lambda_\pm x)}{|x|^{1+\alpha}}$, $M^{\pm}_{KoBoL}(x) = C_\pm \dfrac{w_{1/2}(\lambda_\pm x)}{|x|^{1+\alpha}}$, and $\dfrac{L(ux)}{L(x)} \to 1$ *for any* $u > 0$, *as* $x\to\infty$. *This means that the ratio* $\dfrac{M^{\pm}_{BTS}(x)}{M^{\pm}_{KoBol}(x)}$ *varies regularly with the exponent (v-1/2). Accordingly, if v> 1/2, the* M_{BTS} *is thicker tailed than the* M_{KoBoL}, *while if* $0 < v < 1/2$, the M_{BTS} *is thinner tailed than the* M_{KoBoL}.

(c) *if* $\alpha > -1$, $M_{BTS}(x)$ *is completely monotone, while if* $\alpha < -1$, *it is not.*

Proof. (a) Follows from the fact that $w_v(\lambda_\pm x) \to 1$ as $|x|\to$ o.

b) From (10), we note that $w_v(x) \sim \dfrac{2^{1-v}}{\Gamma(v)}\sqrt{\dfrac{\pi}{2}}\dfrac{e^{-x}}{x^{(1/2)-v}} = \dfrac{2^{1-v}}{\Gamma(v)}\dfrac{\pi}{2}\dfrac{w_{1/2}(x)}{x^{(1/2)-v}}$ as $x\to\infty$. Hence the proof follows from this fact with $L(x) := \dfrac{\pi}{2}\dfrac{2^{1-v}}{\Gamma(v)}(\lambda_\pm)^{v-1/2}$.

(c) Note that $\dfrac{d}{dx}K_v(x) = -K_{v-1}(x) - \dfrac{v}{x}K_v(x)$ [14, p. 148]. Using this fact, if $\beta > v$, then

$$\frac{d}{dx}x^{\beta}K_v(x) = x^{\beta-1}\left[(\beta - v)K_v(x) - xK_{v-1}(x)\right] > x^{\beta-1}\left[(\beta - v - x)K_v(x)\right]$$

which implies that if $0 < x < \beta - v$, then $\dfrac{d}{dx}x^{\beta}K_v(x) \geqslant 0$. Hence we see that if $1 + \alpha < 0$, then $M_{BTS}(x) = x^{v-(1+\alpha)}K_v(\lambda_\pm x)$ increases near zero, so that M_{BTS} cannot be completely monotone on $(0,\infty)$ for any $v > 0$.

Now for $(1 + \alpha) > 0$, let $k(x) = \dfrac{B_v(x)}{x^{1+\alpha}}$, where $B_v(x) := x^v K_v(x)$. Then $k(\cdot) \in C^\infty(0,\infty)$, and $\dfrac{d}{dx}k(x) = -(1+\alpha)x^{-(2+\alpha)}B_v(x) - x^{-(1+\alpha)}xB_{v-1}(x) < 0$, for all $x > 0$, so that we have $\lim_{x\to 0^+}\dfrac{d}{dx}k(x) = -\infty$, and $\lim_{x\to\infty}\dfrac{d}{dx}k(x) = 0$. So we can deduce that $\lim_{x\to 0^+}\dfrac{d^2}{dx^2}k(x) > 0$, which implies that $k(\cdot)$ becomes a convex function on $(0,\infty)$. Thus, inductively, we can deduce that $(-1)^n\dfrac{d^n}{dx^n}k(x) > 0$ for all $n = 1,2...$, so that M_{BTS} is completely monotone for $\alpha > -1$.

Let $X := (X_t)_{t\geq 0}$ be a Levy process with Levy measure M, and for each $t > 0$, let $\Delta X_t = X_t - X_{t-}$. Consider an index : the *Blumenthal-Getoor(BG) index* $\beta(X)$ of X (which goes back to [31]) defined by

$$\beta(X) := \left\{p > 0 : \int_{|x|\leq 1}|x|^p M(dx) < \infty\right\}.$$

We clearly see that if X is an α-stable process, then $\beta(X) = \alpha$. Define another index, the *pth-variation index* γ (X) of X (following terminology of [12,31] by

$$\gamma(X) := \inf\left\{p > 0 : \sum_{0\leq s\leq t}^{|\Delta Xs|\neq 0}|\Delta X_s|^p < \infty \ \ for\ all\ t \geq 0\right\},$$

where if the set is empty, $\gamma(X) = \infty$.

Let us recall that the random measure of jumps of a Levy process X with Levy measure M, denoted by J_X, is a Poisson measure on $[0,\infty) \times \mathbb{R}$ defined by, for any $t \geq 0$ and Borel set $A \subset \mathbb{R}$,

$$J_X(\omega,[0,t]\times A) = \#\{s : s\in[0,t], \Delta X_s(\omega) \neq 0, \Delta X_s(\omega)\in A\}$$

with the intensity measure $E[J_X(\omega,[o,t]\times A)] = tM(A)$.

Proposition 13. (pth-variation) *Let* $X := (X_t)_{t\geq 0} \sim BTS(\alpha, C_\pm, v, \lambda_\pm, \gamma)$. *Let* $\alpha < 2$, *and* $p > 0$.
if $\alpha < p$, *then* $\int_{|x|\leq 1}|x|^p M_{BTS}(dx) < \infty$, *and further it holds that*

$$\sum_{0\leq s\leq t}^{\Delta X_s\neq 0}|\Delta X_s(w)|^p < \infty \text{ for all } t>0, \text{ a.e.}\omega \text{ if and only if } \int_{|x|<1}|x|^p M_{BTS}(dx) < \infty,$$

so that $\beta(X)=\gamma(X)=\alpha$, *a.e.. In particular, if* $p = 1$, *then*

$$\sum_{0\leq s\leq t}^{\Delta X_s\neq 0}|\Delta X_s(w)| < \infty \text{ for all } t>0, \text{ a.e.}\omega \text{ if and only if } \int_{|x|\leq 1}|x|M_{BTS}(dx) < \infty,$$

so that the jump process has the finite first variation if only if $\alpha < 1$.

Proof. Fixed $\in$, $t > 0$ define $S_t^{\in} = \Sigma_{0\leq s\leq t}|\Delta X_s|^p 1_{\{|\Delta X_s| > \in\}}$, Then $S_t^{\in} = \int_0^t\int_{\{|\Delta X_s| > \in\}}|x|_p J_X(ds\times dx)$, and according to [19, Proposition 19.5]], the random variable $S_t^{\in}$ follows a compound Poisson distribution on $\mathbb{R}$ whose characteristic function is given by

$$E\left[e^{iuS_t^{\in}}\right] = \exp\left\{t\int_{|x|>\in}\left(e^{iu|x|^p} - 1\right)M_{BST}(dx)\right\}$$

$$= \exp\left\{t\int_{\in^p}^{\infty}\left(e^{iux} - 1\right)v(dx)\right\}, \quad u\in\mathbb{R}$$

where $\phi(x) := |x|^p$ on $\mathbb{R}$, and $v(dx) := (M_{BTS}\phi^{-1})$. We easily note that $\int_0^\infty|e^{iux} - 1|v(dx) < \infty$ if and only if $\int_0^1|x|v(dx) < \infty$, But if $\alpha < p$, by Lemma 5, $\int_{|x|\leq 1}|x|^p M_{BTS}(dx) < \infty$, so that as $\in\downarrow 0$, $S_t^{\in}$ converges in distribution to a random variable $S_t := \sum_{0\leq s\leq t}^{\Delta X_s\neq 0}|\Delta X_s|^p$ with $E[e^{iuS_t}] = \exp\{t\int_0^\infty(e^{iux} - 1)v(dx)\}$ for all $t \geq 0$. Hence $(S_t)_{t\geq 0}$ becomes a compound

Poisson process. Therefore, **P** [$S_t < \infty$ for all $t \geq 0$]=1 if and only if $\int_0^\infty |e^{iux}-1|\nu(dx) < \infty$, and in turn if and only if $\int_{|x|\leq 1} |x|^p M_{BTS}(dx) < \infty$. This completes the proof of the first part. The proof of the second part follows from Lemma 5 (a).

The following result has been proved by authors for generalized TS processes dened in [32]. For the sake of completeness, we will prove it within the class of our BTS processes.

Theorem 14. (short and long time behavior) Let $\alpha \in (0,2)$ and $(X_t)_{t\geq 0} \sim$ BTS(α, $C_\pm$, ν, $\lambda_\pm$, 0).

(a) For $h \in (0,1)$, let $a_h = -\int_{\{h^{1/\alpha}<|x|<1\}} xM_{BTS}(dx)$. Then, as $h \to 0$, $\{h^{-1/\alpha}(X_{ht}-hta_h)\}_{t\geq 0}$ converges in distribution to an α-stable process $(X_t^\alpha)_{t\geq 0}$, where X_1^α is a strictly α-stable random variable with Levy measure $M_{stable}(x)\,dx := \left(\frac{C_+}{x^{1+\alpha}}1_{x>0}(x)+\frac{C_-}{|x|^{1+\alpha}}1_{x<0}(x)\right)dx$.

(b) *Let* $b_h = -\int_{\{|x|>1\}} x\, M_{BTS}(dx)$, *Then, as* $h \to \infty$, $\{h^{-1/2}(X_{ht}-htb_h)\}_{t\geq 0}$ *converges in distribution to a Brownian motion* $(B_t)_{t\geq 0}$ *with mean 0 and variance* $\int_{-\infty}^{\infty} x^2 M_{BTS}(dx)$.

Proof. (a) For each $t > 0$, put $Y_t^h = h^{-1/\alpha}(X_{ht}-hta_h)$. We first see that by Lemma 5 (a), a_h exists for any $\alpha \in (0,2)$. We next see from (1) that the characteristic function of Y_t^h is given by

$$\phi_{Y_t^h}(u) = \exp\left\{ht\int_{-\infty}^{\infty}\left(e^{iu(h^{-1/\alpha}x)}-1-iu(h^{-1/\alpha}x)1_{\{|h^{-1/\alpha}x|\leq 1\}}(h^{-1/\alpha}x)\right)M_{BTS}(x)\,dx\right\}$$

$$= \exp\left\{ht\int_{-\infty}^{\infty}\left(e^{iuy}-1-iuy1_{\{|y|\leq 1\}}(y)\right)M_{BTS}\left(h^{1/\alpha}y\right)h^{1/\alpha}dy\right\}$$

But $hM_{BTS}\left(h^{1/\alpha}y\right)h^{1/\alpha} = \frac{C_+w_\nu\left(\lambda_+h^{1/\alpha}y\right)}{y^{1+\alpha}}1_{y>0}(y)$

$$+\frac{C_-w_\nu\left(\lambda_-h^{1/\alpha}y\right)}{y^{1+\alpha}}1_{y<0}(y) \to M_{stable}(y), \text{ as } h \to 0.$$

Hence we have that

$$\lim_{h\to 0}\phi_{Y_t^h}(u) = \exp\left\{t\int_{-\infty}^{\infty}\left(e^{iuy}-1-iuy1_{\{|y|\leq 1\}}(y)\right)M_{stable}(y)\,dy\right\},$$

so that the Levy continuity theorem completes the proof of (a).

(b) For each t, $h > 0$, put $Z_t^h = h^{-1/2}(X_{ht}-htb_h)$. Then from (16), it follows that

$$\phi_{Z_t^h}(u) = E[e^{iuh^{-1/2}X_{ht}}].\exp\left\{-ih^{1/2}tu\int_{\{|x|>1\}} xM_{BTS}(dx)\right\}$$

$$= \exp\left\{ht\int_{-\infty}^{\infty}\left(e^{i\left(h^{-1/2}u\right)x}-1-i\left(h^{-1/2}u\right)x\right)M_{BTS}(dx)\right\}$$

$$= \exp\left\{t\int_{-\infty}^{\infty}\left(h\sum_{n=2}^{\infty}\frac{\left(ih^{-1/2}ux\right)^n}{n!}\right)M_{BTS}(dx)\right\}.$$

But when $h \to \infty$, the integrand in the exponent convergres to $-\frac{u^2}{2}x^2$. Hence it follows from the dominated convergence theorem that we have

$$\lim_{h\to\infty}\phi_{Z_t^h}(u) = \exp\left\{-\frac{u^2}{2}t\int_{-\infty}^{\infty} x^2M_{BTS}(dx)\right\},$$

so that the Levy continuity theorem completes the proof of (b).

Equivalence of Measures for BTS Processes

In this section, we shall investigate necessary and suffcient conditions for the equivalence of probability measures on the function space induced by BTS processes.

Let $\Omega := D[0,\infty)$ denote the Skorokhod space of all right continuous function with left limit from $[0,\infty)$ to $\mathbb{R}$. Define $X_t(\omega) = \omega(t)$ for $\omega \in D[0,\infty)$ and $t \geq 0$. Set $\boldsymbol{F}_t = (X_s : s \in [0,t])$ and $\boldsymbol{F} = \sigma(X_t : t \geq 0)$. It is well-known that any Levy process induces a probability measure **P** on (Ω, $\boldsymbol{F}$) under which the canonical process $(X_t)_{t\geq 0}$ is identical in distribution with the Levy process [19]. Thus we may regard any Levy process as a probability measure on the Skorokhod space $D[0,\infty)$ and vice versa.

The following theorem is an immediate consequence of Theorems 33.1 and 33.2 in [19], which determines an equivalent probability measure under which the pure jump Levy process process becomes again a pure jump Levy process.

Theorem 15. Let $(X_t)_{t\geq 0}$ *be a pure jump Levy process whose Levy triplets are* (γ, 0, M) and ($\tilde{\gamma}$, 0, $\tilde{M}$) under measures **P** and **Q** on (Ω, $\boldsymbol{F}$), respectively. Then $\mathbf{P}|_{F_t}$ *and* $Q|_{F_t}$ *are equivalent for every* $t \in (0,\infty)$ *if and only if Levy triplets satisfies the following conditions:*

The Levy measures M and $\tilde{M}$ *are equivalent with*

$$\int_{-\infty}^{\infty}\left(\sqrt{\frac{\tilde{M}(dx)}{M(dx)}}-1\right)^2 M(dx) < \infty, \tag{18}$$

and γ and γ satisfy

$$\tilde{\gamma}-\gamma = \int_{|x|\leq 1} x\left(\tilde{M}-M\right)(dx). \tag{19}$$

We will use this theorem to prove Theorem 17 below, in which the integrability Condition (18) against the Levy measure $M(dx) = M_{BTS}(dx)$ will be critical. Due to (10), this condition is equivalent to the finiteness of the integral over some neighborhood of $x = 0$, so that we need to know the rate of convergence of $w_\nu(|x|)$ as $|x| \to 0$, as will be shown in the following lemma.

Lemma 16. *Let* $w_\nu(x) := \frac{2^{1-\nu}}{\Gamma(\nu)}x^\nu K_\nu(x)$, $x \geq 0$. *Then it holds that:*

(a) *if* $\nu \geq 1/2$, *the right-hand derivative of* w_ν *at* 0 := $(w_\nu)'_+(0)$ *exists, while if* $0 < \nu < 1/2$, *it does not exist.*

(b) $w_\nu(x) = \begin{cases} 1+O(x), & \text{for } \nu \geq 1/2 \\ 1+O\left(x^{2\nu}\right), & \text{for } 0<\nu</2 \end{cases}$, $\text{as } x \to 0+$

Here, O denotes, as usual, the big oh notation to describe the error term.

Proof. (a) Recall that $\frac{d}{dx}(x^\nu K_\nu(x)) = -x^\nu K_{\nu-1}(x)$ for $x > 0$, and w_ν is continuous on $[0,\infty)$. Observe that if $\beta > \nu$, $\lim_{x\to 0+} x^\beta K_\nu(x) = 0$, while if $\beta < \nu$, the limit does not exist. From these facts, it follows that

$$(w_\nu)'_+(0) = \lim_{x\to 0+}\frac{d}{dx}w_\nu(x) := \frac{d}{dx}w_\nu(0+) = \begin{cases} -\infty, & \text{for } 0<\nu<1/2 \\ -1, & \text{for } \nu = 1/2 \\ 0, & \text{for} \nu > 1/2 \end{cases},$$

which proves (a).

(b)If $\nu \geq 1/2$, then by Taylor theorem, it follows that $w_\nu(x) = 1+O(x)$ as $x \to 0+$. On the other hand, if $0 < \nu < 1/2$, then $\frac{d}{dx}w_\nu(0+)$ does not exist, and so we cannot use Taylor theorem to obtain a first-order approximation of $w_\nu(x)$ around $x = 0$. But we can choose a function

$f(x)$ defined on $[0,\infty)$ so that $\frac{d}{dx}(w_\nu + f)(0+)$, exists. In fact, using the recurrence relation: $K_\nu(x) + \frac{xK_{\nu-1}(x)}{2\nu} = \frac{xK_{\nu+1}(x)}{2\nu}$ [14, p.148] we can choose the function $f(x)$ as

$$f(x) := \frac{2^{1-\nu}}{\Gamma(\nu)}\cdot\frac{x^{\nu+1}K_{\nu-1}(x)}{2\nu} = \frac{2^{-\nu}}{\Gamma(\nu+1)}x^{\nu+1}K_{\nu-1}(x)$$

for which $w_\nu(x)+f(x) = w_{\nu+1}(x)$. But we see that since $(w_{\nu+1})'_+(0)$ exists, we have that $w_{\nu+1}(x) = 1+O(x)$ as $x\to 0+$, and also by observing that

$$x^{\nu+1}K_{\nu-1}(x) = x^{2\nu}\left(x^{1-\nu}K_{1-\nu}(x)\right) = O\left(x^{2\nu}\right),\ as\ x\to 0+$$

we see that $f(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)}O(x^{2\nu}) = O(x^{2\nu})$. Accordingly, we have

$$w_\nu(x) = 1 + \left(-f(x)\right) + O(x)$$
$$= 1 + O\left(x^{2\nu}\right),\ as\ x\to 0+,$$

which completes the proof of (b).

Theorem 17. Let $X := (X_t)_{t\geq 0}$ denote the canonical process on $(\mathbf{\Omega}, \boldsymbol{F})$. *Let* $\mathbf{P}$ be a probability measure on $(\mathbf{\Omega}, \boldsymbol{F})$ under which $\mathbf{X} \sim$ BTS $(\alpha, C_\pm, \nu, \lambda_\pm, \gamma)$ *and let Q be another probability measure on* $(\mathbf{\Omega}, \boldsymbol{F})$ *under which* $\mathbf{X} \sim$ BTS $(\tilde{\alpha}, \tilde{C}_\pm, \tilde{\nu}, \tilde{\lambda}_\pm, \tilde{\gamma})$. *Assume that* $\lambda_\pm, \tilde{\lambda}_\pm > 0$. *Then* $\mathbf{P}|\boldsymbol{F}_t$ *and* $\mathbf{Q}|\boldsymbol{F}_t$ *are equivalent for every* $t \in (0,\infty)$ *if and only if the following conditions hold:*

(a) either $\nu, \tilde{\nu} \geq 1/2$ *or* $0 < \nu, \tilde{\nu} < 1/2$ *exclusively.*

(b)if $\nu, \tilde{\nu} \geq 1/2$ *then* $\alpha = \tilde{\alpha}$, $C_\pm = \tilde{C}_\pm$ *and*

$$\tilde{m}-m = \frac{C\Gamma\left(\frac{1-\alpha}{2}\right)}{2^\alpha}\left(\frac{\Gamma\left(\tilde{\nu}+\frac{1-\alpha}{2}\right)}{\Gamma(\tilde{\nu})}\left(\tilde{\lambda}_+^{\alpha-1} - \tilde{\lambda}_-^{\alpha-1}\right) - \frac{\Gamma\left(\nu+\frac{1-\alpha}{2}\right)}{\Gamma(\nu)}\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}\right)\right) \tag{20}$$

and if $0 < \nu, \tilde{\nu} < 1/2$, *then* $\alpha = \tilde{\alpha}$, $C_\pm = \tilde{C}_\pm$, $\nu = \tilde{\nu}$ *and*

$$\tilde{m}-m = \frac{C\Gamma\left(\frac{1-\alpha}{2}\right)}{2^\alpha}\bullet\frac{\Gamma\left(\nu+\frac{1-\alpha}{2}\right)}{\Gamma(\nu)}\left[\left(\tilde{\lambda}_+^{\alpha-1} - \tilde{\lambda}_-^{\alpha-1}\right) - \left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}\right)\right] \tag{21}$$

where $m = \gamma + \int_{|x|>1} xM_{BTS}(dx)$ *and* $\tilde{m} = \tilde{\gamma} + \int_{|x|>1} x\tilde{M}_{BTS}(dx)$.

Moreover, if $\frac{C_+}{\alpha} \neq \frac{\tilde{C}_+}{\tilde{\alpha}}\left(or\ \frac{C_-}{\alpha} \neq \frac{\tilde{C}_-}{\tilde{\alpha}}\right)$, *then* $\mathbf{P}|\boldsymbol{F}_t$ *and* $\mathbf{Q}|\boldsymbol{F}_t$ *are singular for every* $t \in (0,\infty)$.

Proof. Note that Condition (18) for the BTS process can be written as

$$\int_0^\infty\left(\sqrt{\tilde{C}_+}\left(w_{\tilde{\nu}}(\tilde{\lambda}_+x)\right)^{\frac{1}{2}}x^{-\frac{\tilde{\alpha}+1}{2}} - \sqrt{C_+}\left(w_\nu(\lambda_+x)\right)^{\frac{1}{2}}x^{-\frac{\alpha+1}{2}}\right)^2dx$$
$$+\int_{-\infty}^0\left(\sqrt{\tilde{C}_-}\left(w_{\tilde{\nu}}(\tilde{\lambda}_-|x|)\right)^{\frac{1}{2}}|x|^{-\frac{\tilde{\alpha}+1}{2}} - \sqrt{C_-}\left(w_\nu(\lambda_-|x|)\right)^{\frac{1}{2}}|x|^{-\frac{\alpha+1}{2}}\right)^2dx \tag{22}$$

For the proof of theorem, we shall only consider the first term of (22), since the second term can be treated in similar way. If $\alpha < \tilde{\alpha}$, then for any λ_+ and $\tilde{\lambda}_+$, we have

$$\lim_{x\to 0+}\frac{\sqrt{\tilde{C}_+}\left(w_{\tilde{\nu}}(\tilde{\lambda}_+x)\right)^{\frac{1}{2}}x^{-\frac{\tilde{\alpha}+1}{2}} - \sqrt{C_+}\left(w_\nu(\lambda_+x)\right)^{\frac{1}{2}}x^{-\frac{\alpha+1}{2}}}{x^{-\frac{\tilde{\alpha}+1}{2}}} = \sqrt{\tilde{C}_+}.$$

If $\alpha = \tilde{\alpha}$ but $C_+ < \tilde{C}_+$, then for any λ_+ and $\tilde{\lambda}_+$, we have

$$\lim_{x\to 0+}\frac{\sqrt{\tilde{C}_+}\left(w_{\tilde{\nu}}(\tilde{\lambda}_+x)\right)^{\frac{1}{2}}x^{-\frac{\alpha+1}{2}} - \sqrt{C_+}\left(w_\nu(\lambda_+x)\right)^{\frac{1}{2}}x^{-\frac{\alpha+1}{2}}}{x^{-\frac{\alpha+1}{2}}} = \sqrt{\tilde{C}_+} - \sqrt{C_+}.$$

Hence, if either $\alpha < \tilde{\alpha}$ or $\alpha = \tilde{\alpha}$ but $C_\pm < \tilde{C}_\pm$, then the first integrand of (22) is equivalent to $x^{-\alpha-1}$ near zero, and hence the first term of (22) cannot be integrable.

Now let $\alpha = \tilde{\alpha}$ and $C_\pm = \tilde{C}_\pm := C$. Then the first term of (22) is equal to $C\int_0^\infty\left(\sqrt{w_{\tilde{\nu}}(\tilde{\lambda}_+x)} - \sqrt{w_\nu(\lambda_+x)}\right)^2\frac{dx}{x^{1+\alpha}}$.

Now let's find the necessary and sufficient condition for which this integral is finite for all $\alpha \in (0,2)$. To do this, let's first assume that $0 < \nu, \tilde{\nu} < 1/2$. Then we have, by Lemma 16 (b), as $x\to 0+$, $w_\nu(\lambda_+x) = 1 + O(x^{2\nu})$ and $w_{\tilde{\nu}}(\tilde{\lambda}_+x) = 1 + O(x^{2\tilde{\nu}})$. On the other hand, $\sqrt{1+u} = 1 + u/2 + o(u)$ for $|u|\to 0$. Accordingly, it holds that for suffciently small enough $\delta > 0$,

$$\int_0^\delta\left(\sqrt{w_{\tilde{\nu}}(\tilde{\lambda}_+x)} - \sqrt{w_\nu(\lambda_+x)}\right)^2\frac{dx}{x^{1+\alpha}}$$
$$= \frac{1}{4}\int_0^\delta\left(O\left(x^{2\nu}\right) - O\left(x^{2\tilde{\nu}}\right)\right)^2\frac{dx}{x^{1+\alpha}}.$$

Hence this integral has to be finite for all $\alpha \in (0,2)$ if and only if $\nu = \tilde{\nu}$.

Next, let's assume that $\nu, \tilde{\nu} \geq 1/2$. Then we have, by Lemma 16 (b), $w_\nu(x) = 1 + O(x)$, as $x\to 0+$. Therefore, for suffciently small enough $\delta > 0$, we have that

$$\int_0^\delta\left(\sqrt{w_{\tilde{\nu}}(\tilde{\lambda}_+x)} - \sqrt{w_\nu(\lambda_+x)}\right)^2\frac{dx}{x^{1+\alpha}} = \frac{1}{4}\int_0^\delta O\left(x^2\right)\frac{dx}{x^{1+\alpha}} < \infty.$$

for every $\alpha \in (0,2)$. Hence, if $\nu; \tilde{\nu} \geq 1/2$, this integral becomes finite for all $\alpha \in (0,2)$.

In all, $0 < \nu, \tilde{\nu} < 1/2$, Condition (18) holds if and only if $\alpha = \tilde{\alpha}$, and $C_\pm = \tilde{C}_\pm$ and $\nu = \nu$. If $\nu, \tilde{\nu} \geq 1/2$ then Condition (18) holds if and only if $\alpha = \tilde{\alpha}$ and $C_\pm = \tilde{C}_\pm$.

Now, to complete the proof of (a), assume that either $0 < \nu < 1/2$, $\tilde{\nu} \geq 1/2$ or $0 < \tilde{\nu} < 1/2$, $\nu \geq 1/2$. let's say $0 < \nu < 1/2$, $\tilde{\nu} \geq 1/2$. Then there exists an $\alpha \in (0,2)$ such that for any $\delta > 0$,

$$\int_0^\delta\left(\sqrt{w_{\tilde{\nu}}(\tilde{\lambda}_+x)} - \sqrt{w_\nu(\lambda_+x)}\right)^2\frac{dx}{x^{1+\alpha}}$$
$$= \frac{1}{4}\int_0^\delta O\left(x^{4\nu}\right)\frac{dx}{x^{1+\alpha}} = \infty,$$

which implies that Condition (18) cannot be satisfied. This complete the proof of (a).

Now, consider condition (19) for the BTS process. First, let $\nu, \tilde{\nu} \geq 1/2$, $\alpha = \tilde{\alpha}$, and $C_\pm = \tilde{C}_\pm$. Then we see that the integral

$$\int_{|x|>1} x\left(\tilde{M}_{BTS} - M_{BTS}\right)(dx)$$

exists due to Proposition 6, and that since from Condition (18) it follows that the integral

$$\int_{|x|\leq 1} x\left(\tilde{M}_{BTS} - M_{BTS}\right)(dx)$$

is finite (see Remark 33.3 in [19]). Hence the integral

$$\int_{-\infty}^\infty x\left(\tilde{M}_{BTS} - M_{BTS}\right)(dx) =$$
$$\int_0^\infty x\left(\tilde{M}_{BTS} - M_{BTS}\right)(dx) + \int_{-\infty}^0 x\left(\tilde{M}_{BTS} - M_{BTS}\right)(dx)$$

exists, Indeed, we have, by condition $C_+ = \tilde{C}_+ := C$,

$$\int_0^\infty x\left(\tilde{M}_{BTS} - M_{BTS}\right)(dx) = C\int_0^\infty x^{-\alpha}\left(w_{\tilde{\nu}}(\tilde{\lambda}_+x) - w_\nu(\lambda_+x)\right)dx$$

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$$= C\int_0^\infty \tilde{\lambda}_+^{\alpha-1}\frac{2^{1-\tilde{v}}}{\Gamma(\tilde{v})}\left(\tilde{\lambda}_+ x\right)^{\tilde{v}-\alpha}K_{\tilde{v}}\left(\tilde{\lambda}_+ x\right)d\left(\tilde{\lambda}_+ x\right)$$

$$-C\int_0^\infty \lambda_+^{\alpha-1}\frac{2^{1-v}}{\Gamma(v)}\left(\lambda_+ x\right)^{v-\alpha}K_v\left(\lambda_+ x\right)d\left(\lambda_+ x\right)$$

$$= C\tilde{\lambda}_+^{\alpha-1}\frac{2^{1-\tilde{v}}}{\Gamma(\tilde{v})}\int_0^\infty z^{\tilde{v}-\alpha}K_{\tilde{v}}(z)\,dz - C\lambda_+^{\alpha-1}\frac{2^{1-v}}{\Gamma(v)}\int_0^\infty z^{v-\alpha}K_v(z)\,dz$$

$$= C\frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{2^\alpha}\left(\frac{\tilde{\lambda}_+^{\alpha-1}\Gamma\left(\tilde{v}+\frac{1-\alpha}{2}\right)}{\Gamma(\tilde{v})} - \frac{\lambda_+^{\alpha-1}\Gamma\left(v+\frac{1-\alpha}{2}\right)}{\Gamma(v)}\right), \text{ using (15).}$$

Using the same arguments as above, we can obtain

$$\int_{-\infty}^0 x\left(\tilde{M}_{BTS}-M_{BTS}\right)(dx)$$

$$= -\frac{C\Gamma\left(\frac{1-\alpha}{2}\right)}{2^\alpha}\left(\frac{\tilde{\lambda}_-^{\alpha-1}\Gamma\left(\tilde{v}+\frac{1-\alpha}{2}\right)}{\Gamma(\tilde{v})} - \frac{\lambda_-^{\alpha-1}\Gamma\left(v+\frac{1-\alpha}{2}\right)}{\Gamma(v)}\right)$$

Combining two integrals above, we obtain

$$\int_{-\infty}^\infty x\left(\tilde{M}_{BTS}-M_{BTS}\right)(dx) \tag{23}$$

$$= \frac{C\Gamma\left(\frac{1-\alpha}{2}\right)}{2^\alpha}\left(\frac{\Gamma\left(\tilde{v}+\frac{1-\alpha}{2}\right)}{\Gamma(\tilde{v})}\left(\tilde{\lambda}_+^{\alpha-1}-\tilde{\lambda}_-^{\alpha-1}\right) - \frac{\Gamma\left(v+\frac{1-\alpha}{2}\right)}{\Gamma(v)}\left(\lambda_+^{\alpha-1}-\lambda_-^{\alpha-1}\right)\right)$$

Since $\gamma = m - \int_{|x|>1} xM_{BTS}(dx)$, and $\tilde{\gamma} = \tilde{m} - \int_{|x|>1} x\tilde{M}_{BTS}(dx)$, it follows that

$$\tilde{\gamma}-\gamma = \tilde{m}-m-\int_{|x|\ge 1} x\left(\tilde{M}_{BTS}-M_{BTS}\right)(dx)$$

$$= \tilde{m}-m-\left(\int_{-\infty}^{\infty} x\left(\tilde{M}_{BTS}-M_{BTS}\right)(dx) - \int_{|x|\le 1} x\left(\tilde{M}_{BTS}-M_{BTS}\right)(dx)\right)$$

Rearranging this, we have

$$\tilde{\gamma}-\gamma-\int_{|x|\le 1} x\left(\tilde{M}_{BTS}-M_{BTS}\right)(dx)$$

$$= \tilde{m}-m-\left(\int_{-\infty}^{\infty} x\left(\tilde{M}_{BTS}-M_{BTS}\right)(dx)\right)$$

Therefore, from (23), it follows that Condition (19) holds if and only if Equation (20) is satisfised.

Now, next consider the case: $0 < v$, $\tilde{v} < 1/2$, $\alpha = \tilde{\alpha}$, $C_\pm = \tilde{C}_\pm$ and $v = \tilde{v}$. Then from (23) again, it follows that Condition (19) holds if and only if Equation (21) is satisfied.

Finally, it remains to prove the last statement of the theorem. For fixed $t > 0$, define a sequence of random variables $\{Y_n^t\}_{n=1}^\infty$ by

$$Y_n^t = \#\{s : s \in [0,t], \Delta X_s \in [(n+1)^{-1/\alpha}, n^{-1/\alpha}]\}, n \in N.$$

Then $\{Y_n^t\}$'s are mutually independent and are Poisson distributed with mean :

$$E[Y_n^t] = tC_+\int_{(n+1)^{-1/\alpha}}^{n^{-1/\alpha}}\frac{w_v\left(\lambda_+ x\right)}{x^{1+\alpha}}\,dx.$$

By Lemma 16 (b), we see that $w_v(\lambda_+ x) = 1 + o(1)$ as $x \downarrow 0$, Hence it holds that

$$\lim_{n\to\infty} E[Y_n^t] = t\frac{C_+}{\alpha}$$

which implies that the Cesaro mean $\frac{1}{n}\sum_{j=1}^n E\left[Y_j^t\right]$ should converge to $t\frac{C_+}{\alpha}$ as $n\to\infty$. Furthermore, since $E\left[Y_n^t\right] = Var\left[Y_n^t\right]$ for each n, and $\left\{Var\left[Y_n^t\right]\right\}_{n=1}^\infty$ is bounded, it follows that

$$\sum_{n=1}^\infty \frac{Var[Y_n^t]}{n^2} < \infty .$$

Therefore, by Kolmogorov's strong law of large numbers, it holds that

$$P\left[\lim_{n\to\infty}\frac{S_n^t}{n} = t\frac{C_+}{\alpha}\right] = 1$$

where $S_n^t := \sum_{j=1}^n E[Y_j^t]$. Hence, if $\frac{C_+}{\alpha} \ne \frac{\tilde{C}_+}{\tilde{\alpha}}$, then we have $\mathbf{P}(A^t) = 1$, but $\mathbf{Q}(A^t) = 0$, where $A^t := \left\{\omega : \lim_{n\to\infty}\frac{S_n^t(\omega)}{n} = t\frac{C}{\alpha}\right\} \in F_t$ for $t > 0$. Therefore $\mathbf{P}|F_t$ *and* $\mathbf{Q}|F_t$ are singular for every $t \in (0, \infty)$.

The following corollary is a simple consequence of Theorem 17 and Condition (19) (cf, [32, Theorem 4.1]).

Corollary 18. Under the hypotheses of Theorem 17 with being $\lambda_\pm = 0$, $\tilde{\lambda}_\pm > 0$, if $\tilde{v} \ge 1/2$, then $\mathbf{P}|F_t$ *and* $\mathbf{Q}|F_t$ are equivalent for every $t \in (0,\infty)$ if and only if $\alpha = \tilde{\alpha}$, $C = \tilde{C}$ and

$$\tilde{\gamma}-\gamma = \int_{|x|<1} x\left(\tilde{M}_{BTS}-M_{stable}\right)(dx).$$

However, the BTS process with $\tilde{v}<1/2$ cannot be equivalent to any stable processes.

Bessel Tempered Stable Model and Option Pricing

In this section we describe our stock price model as an exponential BTS process under both the market and the risk-neutral measure, and then discuss how to calibrate option pricing model consistent with the observed market option prices.

The Stock Price Model

In this subsection we employ the BTS process to model the dynamics of log stock prices. The BTS process have parameters (α, $C_\pm$, v, $\lambda_\pm$, γ) which includes an additional new parameter v which does not exist in the existing tempered stable processes with 4 parameters (α, $C_\pm$, $\lambda_\pm$, γ) such as CGMY and MTS processes. This new parameter v determines the shape of tempering function which enable us to control the jump intensity of the BTS distribution to match the observed jump behavior of log stock prices. The merit of the parameter v is that the shape of tempering function can be chosen by estimating v from the observed market data, unlike in the case of the known tempered stable model whose the tempering function is specified in advance.

By calibrating parameters (α, $C_\pm$, v, $\lambda_\pm$, m) of the BTS distribution from time series market data set over the time interval $[0, \Upsilon]$, in which a stock has been traded, we can model the stock price process as the exponential BTS process $(S_t)_{t\ge 0}$ under the market measure P:

$$S_t = S_0 e^{mt+X_t^0},\ 0 \le t \le \Upsilon$$

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where $m > 0$, and (X_t^0) is the BTS process with parameter $(\alpha, C_\pm, \nu, \lambda_\pm, 0)$. The stock price process $S_t = S_0 e^{mt + X_t^0}$ will be referred to as *exponential BTS model* under **P** or *statistical stock price process*.

The Arbitrage-free Option Price Model

In this subsection, we first show that our model is arbitrage-free: there exists at least one equivalent martingale measure **Q** to **P** such that $(e^{-rt}S_t)_{t\geq 0}$ is a **Q** -martingale where $r > 0$ is the risk neutral interest rate. To show this, we only consider the case where $S_t = S_0 e^{mt + X_t^0}$, under P with $0 < \nu < 1/2$, because the case where $\nu \geq 1/2$ can be treated in the same way. By Theorem 17, there exists an equivalent measure **Q** to **P**, under which $X \sim BTS\,(\alpha, C_\pm, \nu, \lambda_\pm, m)$ becames a $X \sim BTS\,(\tilde{\alpha}, \tilde{C}_\pm, \tilde{\nu}, \tilde{\lambda}_\pm, \tilde{m})$ if and only if $\alpha = \tilde{\alpha}$, $C_\pm = \tilde{C}_\pm$, $\nu = \tilde{\nu}$, and $\tilde{m}$ and m should satisfy Equation (21). So, under this **Q,** S_t is described as $S_0 e^{\tilde{m}t + \tilde{X}_t^0}$, where $\tilde{X}_t^0 \sim (\tilde{\alpha}, \tilde{C}_\pm, \tilde{\nu}, \tilde{\lambda}_\pm, 0)$. In order that $(e^{-rt}S_t)_{t\geq 0}$ be a martingale under Q, $E_Q[e^{-rt}S_t]$ should exist. But, by Theorem 8 $E_Q[e^{\tilde{X}_t^0}] = \phi_{BTS(\alpha, C_\pm, \nu, \tilde{\lambda}_\pm, 0)}(-i)$, exists if and only if $\frac{1}{\left(\tilde{\lambda}\pm\right)^2} < 1$. So, if $\tilde{\lambda}_\pm > 1$, then $E_Q\,[e^{-rt}S_t]$ exists, and we see that

$$E_Q\left[e^{-rt}S_t\right] = E_Q\left[e^{-rt}S_0 e^{\tilde{m}t + \tilde{X}_t^0}\right] = S_0 e^{(-r + \tilde{m} + \tilde{w})t},$$

where $\tilde{w} = \log E_Q[e^{\tilde{X}_1^0}]$. Hence, $(e^{-rt}S_t)_{t\geq 0}$ is a **Q**-martingale if and only $-r + \tilde{m} + \tilde{w} = 0$. Therefore, $\tilde{\lambda}_\pm > 1$, and $\tilde{m} = r - \tilde{w}$, then $E_Q[e^{-rt}S_t] = S_0$, which means that our model is arbitrage-free. Under the condition: $\tilde{\lambda}_\pm > 1$, the stock price process $S_t = S_0 e^{(r - \tilde{w})t + \tilde{X}_t^0}$ will be referred to as the *exponential BTS model under Q, or risk neutral stock price process.*

Now we next show that there are infinitely many equivalent martingale measures, so that our model, *i.e.* the market is incomplete : We note that Equation (21) is a single equation with two unknown free parameters, $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$. Therefore, the equation have infinitely many solutions, so that we are unable to have the unique equivalent martingale measure to the market measure **P**, which means that the market is incomplete.

We finally calibrate an arbitrage-free option model which is consistent with a set of the observed market prices of call options on the stock for strikes and maturities $0 < T \leq \Upsilon$: As mentioned above, since there are infinitely many martingale measures to the market measure P, we need to select a suitable one from among them by calibrating model parameters $\tilde{\nu}$, $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ to the set of the observed market option prices. This can be done by minimizing the least square error between the observed market option prices [14, 6.3] and the model option price obtained, through the fast Fourier transform, from the characteristic function of risk-neutral stock price process in Theorem 8 [14, 2.5.2]) satisfying the following conditions:

if $0 < \nu < 1/2$, then $\alpha = \tilde{\alpha}$, $C = \tilde{C}$, $\tilde{\nu} = \nu$, and

$$r - m = \tilde{\omega} + \frac{C\Gamma\left(\frac{1-\alpha}{2}\right)}{2^\alpha}\frac{\Gamma\left(\nu + \frac{1-\alpha}{2}\right)}{\Gamma(\nu)}\left(\tilde{\lambda}_+^{\alpha-1} - \tilde{\lambda}_-^{\alpha-1} - \left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}\right)\right).$$

Competing Interests

The authors declare that they have no competing interests.

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