Optimal Multi-period Mean-Variance Portfolio Selection for Time Series Return Process

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Abstract

The mean-variance formulation by Markowitz in the 1950s paved a foundation for modern portfolio selection analysis in single period. The analytical optimal solution to the mean-variance formulation in multiperiod portfolio selection has been considered. However, the return process of the portfolio are still assuming to be i.i.d. processes. In this paper, we consider optimal mean-variance portfolio problem in multiperiod with time series return processes. An analytical optimal solution is derived by dynamic programming to maximize an utility function of the expected value and the variance of the terminal wealth. The derived analytical optimal solution is expressed by expected value and variance of time series return processes. Therefore, we can observe the time series effect on the optimal solution of multiperiod portfolio.

Mean-variance Formulation Frontier and Dynamic Programming

The mean-variance formulation for modern portfolio selection analysis in a single period have been widely developed (see e.g. Sharpe et al. [1]). In the i.i.d. setting, the analytical optimal solution to the mean-variance formulation in multiperiod portfolio selection has been also considered by many authors (see e.g. Li et al. [2], Li and Ng[3] and Samuelson[4]). In this section, we consider a capital market with with \((n + 1)\) risky securities, with random rate of returns. An investor joins the market at time 0 an initial wealth \(x_0\). The investor can allocate among the \((n + 1)\) assets. The rate of risky securities at time period \(t\) are denoted by a vector \(\{e_t = e_{01}, e_{02}, \ldots, e_{0n}\}\), where \(e_t\) is random return for securities at time period \(t\) are denoted by a vector. Return \(e_t\) has a known mean \(E(e_t)\) has a known mean \(E(e_t) = E(e_{01}), E(e_{02}), \ldots, E(e_{0n})\) and a known covariance

\[
\text{cov}(e_t) = \begin{pmatrix}
\sigma_{t,00} & \cdots & \sigma_{t,0n} \\
\sigma_{t,0n} & \ddots & \sigma_{t,nn} \\
\vdots & \ddots & \vdots \\
\sigma_{t,nn} & \cdots & \sigma_{t,00}
\end{pmatrix}
\]

Let \(x_t\) be the wealth of investor at the beginning of the \(t\) the period, and let \(u_t, t = 1, 2, \ldots, n\) be amount invested in the \(i\) th risky asset at beginning of the \(t\) time period. The amount invested in the \(0\) th risky asset at the beginning of the \(t\) time period is equal to \(x_t = -\sum_{i=1}^{n} u_t^i\). An investor is seeking a best investment strategy, \(u_t = [u_t^1, u_t^2, \ldots, u_t^n]'\) for \(t = 0, 1, 2, \ldots, T-1\), such that (i) the expected value of the terminal wealth \(x_T\), is maximized if the variance of terminal wealth, \(\text{Var}(x_T)\), is not greater than prescribed risk level, or (ii) the variance of terminal wealth, \(\text{Var}(x_T)\), is maximized if expected terminal wealth, \(E(x_T)\), is not smaller than a prescribed level. Mathematically, a mean-variance formulation for multiperiod portfolio selection can be posed as one of following two forms:

\[
(P1(\sigma)) : \max E(x_T) \\
\text{s.t } \text{Var}(x_T) \leq \sigma
\]

\[
x_{t+1} = \sum_{i=1}^{n} e^i u^i_t + (x_t - \sum_{i=1}^{n} u^i_t)e^i_t \\
= e^0_t x_t + p^i u^i_t \quad t = 0, 1, \ldots, T-1
\]

\[
(P2(\varepsilon)) : \min \text{Var}(x_T) \\
\text{s.t } E(x_T) \geq \varepsilon
\]

Where

\[
P_t = [p_t^1, p_t^2, \ldots, p_t^n]' = [(e_t^1 - e_t^0), (e_t^2 - e_t^0), \ldots, (e_t^n - e_t^0)]'.
\]

Notice that \(E(e_t e_t^i) = \text{Cov}(e_t) + E(e_t)E(e_t^i)\). We assume that \(E(e_t e_t^i)\) is positive definite for all time periods, that is,

\[
E(e_t e_t^i) = \text{E}(\{e_t e_t^i\}^2) > 0 \quad (4)
\]

The following holds from equation (4):

\[
\begin{bmatrix}
\text{E}\{e_t e_t^i\}^2 \\
\text{E}(e_t^0 P_t) \\
\text{E}(P_t e_t^i)
\end{bmatrix} > 0 
\]

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
-1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & 0 & \cdots & 1
\end{bmatrix}
\]

\(\forall t = 0, 1, \ldots, T-1\)

References

Furthermore, we have the following from equation (5):
\[ E(P_t P_t') > 0, \quad \forall t = 0, 1, \ldots, T-1 \] (6)
and
\[ E((e_t^0)^2) - E(e_t^0 P_t) E(P_t P_t') E(e_t^0 P_t') > 0 \quad \forall t = 0, 1, \ldots, T-1 \] (7)

An equivalent formulation to either (P1(\sigma)) or (P2(\sigma)) in generating efficient multiperiod portfolio policies is
\[ (E(\omega)): \max E(x_t) - \text{cov}(x_t) \]
\[ s.t. x_t = e_t^0 x_t + P_t u_t, \quad t = 0, 1, 2, \ldots, T-1 \] (8)

All three problems (P1(\sigma)), (P2(\sigma)), and (E(\omega)) are difficult to solve directly. The optimal multiperiod portfolio policy for problem (E(\omega)) will be derived. The solution to problem (P1(\sigma)) and (P2(\sigma)) will then be obtained based on relationships between (P1(\sigma)), (P2(\sigma)), and (E(\omega)). Define \( \Pi_t(\omega) \) to be the set of optimal solutions of problem (E(\omega)) with given \( \omega_t \), that is
\[ \Pi_t(\omega) = \{ \pi \mid \pi \text{ is maximizer of } (\Lambda(E, \omega_t)) \}. \] (9)

Define
\[ U(E(x_t^0)), E(x_t)) \]
\[ = E(x_t) - \text{Var}(x_t) \]
\[ = -\omega E(x_t^0) + \omega E^2(x_t) + E(x_t) ] \] (10)

It is obvious that \( \tilde{U} \) is a convex function of \( E(x_t) \) and \( E(x_t) \). The following auxiliary problem is now constructed for \( E(\omega) \),
\[ (A(\alpha, \omega)): \max E{-\omega x_t^0 + \alpha x_t} \]
\[ s.t. x_t = e_t^0 x_t + P_t u_t, \quad t = 0, 1, 2, \ldots, T-1. \] (11)

Define \( \Pi_t(t, \omega) \) to be the set of optimal solutions of problem (A(\alpha, \omega)) with given \( \alpha \) and \( \omega \), that is
\[ \Pi_t(\alpha, \omega) = \{ \pi \mid \pi \text{ is maximizer of } (\Lambda(E, \omega_t)) \}. \] (12)

Denote
\[ d(\pi, \omega) = \frac{\partial U(E(x_t^0)), E(x_t))}{\partial E(x_t)} \]
\[ = 1 + 2 \omega E(x_t^0) \] (13)

Now we can introduce the following results (see also Reid[5])

**Lemma 1:** For any positive \( \pi^* \in \Pi_t(\omega) \), \( \pi^* \in \Pi_t(d(\pi^*, \omega_t)) \).

**Lemma 2:** Assume \( \pi^* \in \Pi_t(t, \omega) \). A necessary condition for \( \pi^* \in \Pi_t(t, \omega) \) is \( \lambda^* = 1 + 2 \omega E(x_t) \).

The optimal solution of auxiliary problem (A(\alpha, \omega)) can be derived analytically using dynamic programming. The dynamic programming algorithm starts from stage \( T-1 \). For given \( x_{t-1}^* \), the optimization problem is given as follows:
\[ \max \lambda u_{t-1}(x_{t-1}) \]
\[ = \max \{ \omega E(e_t^0 x_t) + \lambda x_t^0 \}
\[ + \lambda E(P_t P_t') x_{t-1}^0 - 2 \omega x_{t-1}^0 E(e_t^0 P_t u_{t-1}) - \omega E(x_t) - E(\epsilon_t^0 P_t') u_{t-1} \} \]
\[ \text{Optimal } u_{t-1} \text{ can be obtained by solving } \frac{dU(u_{t-1}, x_{t-1})}{du_{t-1}} = 0 \] with
\[ u_{t-1}^* = E^{-1}(P_t P_t') \{ E(P_t P_t') \lambda - E(e_t^0 P_t u_{t-1}) \} \] (15)

Substituting \( u_{t-1}^* \) back to \( J_t(\pi_t(x_{t-1})) \), we have the optimal cost-to-go at given \( x_{t-1} \):
\[ J_t(\pi_t(x_{t-1})) = -\omega E((\epsilon_t^0 x_t)^2) + E(e_t^0 P_t u_{t-1}) E(P_t P_t') E(x_t) + \lambda E(P_t P_t') \{ E(x_t) - E(x_t) - E(P_t P_t') E(x_t) \} ] \]
\[ \lambda^2 E(P_t P_t') E(x_t) \] (16)

The derived utility function has a similar form at stage \( t, 0 \leq t \leq T-1 \), to the original utility function has a similar form at staget \( T \). We can derive the optimal portfolio decision and the optimal cost-to-go for given \( x_t \) at stage \( t, 0 \leq t \leq T-2 \), in a similar manner,
\[ u_t^*(x_t) = E^{-1}(P_t P_t') E(P_t) \]
\[ \frac{\lambda E(P_t P_t')}{2 \omega E(P_t P_t')} \]
\[ \{ E(e_t^0 x_t) + E(e_t^0 P_t u_{t-1}) E(P_t P_t') E(x_t) + \lambda E(P_t P_t') \{ E(x_t) - E(x_t) - E(P_t P_t') E(x_t) \} ] \]
\[ \lambda^2 E(P_t P_t') E(x_t) \} ] \]
\[ = E^{-1}(P_t P_t') E(P_t) \] (17)

Analytical Solution for time series

Time series return process in econometric modeling have been considered mainly in signal period portfolio selection problem (see e.g. Gourieroux[6] and Gourieroux and Jasiak[7]. In this section, we consider the optimal portfolio policy auxiliary problem (A(\lambda, \omega)) at each time period t is of the following form
\[ u_t^*(x_t, \gamma_t) = -K_t x_t + \gamma_t \] (19)

where
\[ K_t = E^{-1}(P_t P_t') E(e_t^0 P_t) \]
\[ \gamma_t = \frac{\lambda^2}{2 \omega} E(P_t P_t') E(P_t) \] (20)

\[ u_t^*(x_t, \gamma_t) = \frac{\lambda}{2} E^{-1}(P_t P_t') E(P_t) \] (21)

\[ \gamma_t = \gamma \langle \frac{\lambda}{2} \rangle_1 E^{-1}(P_t P_t') E(P_t) \] (22)

\[ A_1^t = E(e_t^0 P_t) E(P_t P_t') E(e_t^0 P_t) \] (23)

\[ A_2^t = E(e_t^0 P_t) E(P_t P_t') E(e_t^0 P_t) \] (24)

with the following boundary condition
\[ u_1^*(t, y) = \frac{\lambda}{2} E^{-1}(P_1 P_1') E(P_1) \] (25)

Wealth of investor is expressed as recursive from substituting \( u^*_t \) into \( x_{t+1} \)
\[ x_{t+1}^* = (e_t^0 - P_t K_t) x_t + P_t \gamma_t \] (26)

Squared on both sides of (26) yields
\[ x_{t+1}^2 = [e_t^0 X_t + P_t K_t + k_t P_t K_t X_t^2 (\gamma_t)] \]
\[ + 2(e_t^0 - P_t X_t (\gamma_t) P_t \gamma_t + \gamma_t (P_t P_t') \gamma_t) \] (27)
Then, we take expected values and substitute time series return process to get time series effect on the optimal multiperiod portfolio.

First, we consider MA(1) model

$$P_t = P_{t-1} + B P_{t-1} + \mu_t,$$

(28)

We assume $E(\tilde{P}_t) = 0$, $\tilde{P}_t$ are mutually independent, so that

$$E(\tilde{P}_t, \tilde{P}_j) = E(\tilde{P}_t), \quad i \neq j,$$

(29)

$x_t$ is $f_t$-measurable and $P_t$ is independent of $f_t$ ($P_t$ is not dependent).

We take expectations on both sides of equation (26)

$$E(x_t, (\gamma)) = E((\epsilon_t^0 - P_t^i K_x) x_t(\gamma) + P_t^i v_t(\gamma))$$

(30)

Taking expectation on both sides of (27), we have

$$E(x_t^2, (\gamma)) = E((\epsilon_t^0 - P_t^i K_x) x_t(\gamma) + P_t^i v_t(\gamma))^2 + \epsilon_t^0 P_t^i v_t(\gamma)$$

Taking expectation on both sides of (27), we have

$$E(x_t^2, (\gamma)) = E((\epsilon_t^0 - P_t^i K_x) x_t(\gamma) + P_t^i v_t(\gamma))^2 + \epsilon_t^0 P_t^i v_t(\gamma)$$

(31)

Substituting time series $P_t = \tilde{P}_t + B \tilde{P}_t + \mu_t$, we have

$$E(x_t, (\gamma)) = E(\tilde{P}_t x_t + \tilde{P}_t \tilde{P}_t + \mu_t x_t + \mu_t \tilde{P}_t, (\gamma))$$

(32)

$$E(x_t^2, (\gamma)) = E((\epsilon_t^0 - P_t^i K_x) x_t(\gamma) + P_t^i v_t(\gamma))^2 + \epsilon_t^0 P_t^i v_t(\gamma)$$

(33)

Here, $E(\tilde{P}_t x_t)$ and $E(\tilde{P}_t^2 x_t)$ are unknown. Let $x_t = \tilde{P}_t x_t, z_t = \tilde{P}_t x_t$. Then, we constitute recurrence relation of matrix from $x_t, y_t, z_t$.

Denote

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} H_t & F_t & 0 \\ G_t & J_t & 0 \\ N_t & M_t & 0 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} \delta_{t-1} \\ s_{t-1} \\ \sigma_{t-1} \end{pmatrix},$$

(34)

Where

$$\begin{align*}
H_t &= \epsilon_t^0 - P_t + \mu_t K_x, \\
F_t &= B K_x, \\
\delta_t &= \mu_t \tilde{P}_t,
\end{align*}$$

(35)

$$\begin{align*}
G_t &= (\epsilon_t^0 - P_t + \mu_t K_x) J_t, \\
J_t &= B K_t \tilde{P}_t, \\
\sigma_t &= \mu_t \tilde{P}_t.
\end{align*}$$

(36)

Using (35) and (36), we get

$$\begin{align*}
x_t &= \begin{pmatrix} H_t & F_t & 0 \\ G_t & J_t & 0 \\ N_t & M_t & 0 \end{pmatrix} x_{t-1} + \begin{pmatrix} \delta_{t-1} \\ s_{t-1} \\ \sigma_{t-1} \end{pmatrix},
\end{align*}$$

(37)

Taking expectation and simplifying, we have

$$\begin{align*}
\delta_{t-1} &= \begin{pmatrix} \tilde{P}_t v_t + \tilde{P}_t B v_t + \mu_t v_t \end{pmatrix} \\
\sigma_{t-1} &= \begin{pmatrix} \tilde{P}_t v_t + \tilde{P}_t B v_t + \mu_t v_t \end{pmatrix}.
\end{align*}$$

(38)

Next, we consider MA(2) model

$$P_t = \tilde{P}_t + B \tilde{P}_t + B^2 \tilde{P}_t.$$  

Let $y_t, y_{t-1}, z_t, z_{t-1}$ be the following form

$$\begin{align*}
y_t &= \tilde{P}_t x_t, \\
y_{t-1} &= \tilde{P}_t x_{t-1}, \\
z_t &= \tilde{P}_t x_t, \\
z_{t-1} &= \tilde{P}_t x_{t-1},
\end{align*}$$

(39)

These are necessary to solve optimal solution with MA(2) model.

Then, similarly in MA(1) case, we can express the following matrix form by $x_t, y_t, z_t$ and $z_t^t$. 

$$\begin{align*}
E(\tilde{P}_t x_t) &= \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}.
\end{align*}$$

(40)

Further Discussion

In this section, we consider the general MA(t) model

$$P_t = \tilde{P}_t + B \tilde{P}_t + ... + B^t \tilde{P}_t.$$  

Substituting $P_t = \sum_{i=0}^{\infty} B^i \tilde{P}_t$ into $x_t$, we have

$$\begin{align*}
x_{t+1} &= (\epsilon_t^0 - K_t P_t) x_t + \tilde{P}_t v_t \\
&= (\epsilon_t^0 - K_t \sum_{i=0}^{\infty} B^i \tilde{P}_t) x_t + \tilde{P}_t v_t. \\
&= (\epsilon_t^0 - K \sum_{i=0}^{\infty} B^i \tilde{P}_t) x_t + \tilde{P}_t v_t.
\end{align*}$$

(41)

We need $E(\tilde{P}_t x_t)$ and $E(\tilde{P}_t v_t)$ to solve the optimal solution of multiperiod portfolio. Let $y_t$ and $z_t^t$ be given by the following equations

$$\begin{align*}
y_t &= \tilde{P}_t x_t, \\
z_t &= \tilde{P}_t x_t.
\end{align*}$$

(42)

Then, we see that
xt-1 , yt-k transform this matrix into expression with independent coefficients, like substituting MA(1) model or MA (2) model. So we need to derive in this paper.

We also have to consider the numerical study to demonstrate the efficiency of the solution methods and the adoption of the multiperiod mean-variance formulations for time series return processes. This problem will be left for the further consideration.

The authors declare that they have no competing interests.

Author Contributions

All the authors substantially contributed to the study conception and design as well as the acquisition and interpretation of the data and drafting the manuscript.

References


\[ x_t = (c_{t-1} - K_{t-1}) \begin{pmatrix} \hat{P}_{t-1} \\ \vdots \\ \hat{P}_0 \end{pmatrix} + \begin{pmatrix} \sum_{i=0}^{l-1} B_i y_{t-i} \\ \vdots \\ \sum_{i=0}^{l-1} B_i z_{t-i} \end{pmatrix} + \begin{pmatrix} \hat{P}_{t-1} \\ \vdots \\ \hat{P}_0 \end{pmatrix} \]

\[ (44) \]

\[ \sigma^2 = (v_t \sum_{i=0}^{l-1} B_i P_{t-i} \). \]

Taking expectation and simplifying, we have

\[ E(\sigma^2) = E(v_t \sum_{i=0}^{l-1} B_i P_{t-i} = v_t E(B_{t-i} \hat{P}_{t-i} P_{t-i})). \]

\[ (46) \]

In this case, we see that

\[ x_t = \begin{pmatrix} \hat{P}_{t-1} \\ \vdots \\ \hat{P}_0 \end{pmatrix} \begin{pmatrix} \hat{P}_{t-1} \cdot \hat{P}_{t-2} \cdot \hat{P}_0 \end{pmatrix} \]

\[ = (c_{t-1} - \hat{P}_{t-1} \cdot K_{t-1}) \begin{pmatrix} \hat{P}_{t-1} \cdot \hat{P}_{t-2} \cdot \hat{P}_0 \end{pmatrix} \]

\[ (47) \]

\[ \sum_{i=1}^{l} B_i y_{t-i} \hat{P}_{t-i} \cdot \hat{P}_{t-i} \cdot \hat{P}_0 \]

\[ -K_{t-1} \]

\[ \sum_{i=1}^{l} B_i y_{t-i} \hat{P}_{t-i} \cdot \hat{P}_0 \]

\[ + \begin{pmatrix} \delta_{(1,1)} \\ \vdots \\ \delta_{(l,l)} \end{pmatrix} \]

Now we can not transform this matrix into easy to solve expression like substituting MA(1) model or MA (2) model. So we need to transform this matrix into expression with independent coefficients, \( x_{t-1}, y_{t-1} \) and \( Z_{t-1} \).

This problem will be left for the further consideration.

We also have to consider the numerical study to demonstrate the adoption of the multiperiod mean-variance formulations for time series return processes and the efficiency of the solution methods derived in this paper.

We would like to derive optimal solution case that we have an additional restriction, namely causality from another index to apply our method to pension investment problem, which will be also the further work.