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Optimal Multiperiod Mean-Variance Portfolio Selection for Time Series Return Process

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Abstract

The mean-variance formulation by Markowitz in the 1950s paved a foundation for modern portfolio selection analysis in single period. The analytical optimal solution to the mean-variance formulation in multiperiod portfolio selection has been considered. However, the return process of the portfolio are still assuming to be i.i.d processes. In this paper, we consider optimal mean-variance portfolio problem in multiperiod with time series return processes. An analytical optimal solution is derived by dynamic programming to maximize an utility function of the expected value and the variance of the terminal wealth. The derived analytical optimal solution is expressed by expected value and variance of time series return processes. Therefore, we can observe the time series effect on the optimal solution of multiperiod portfolio.

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Mean-variance Formulation Frontier and Dynamic Programming

The mean-variance formulation for modern portfolio seletion analysis in a single period have been widely developed (see e.g Sharpe et al. [1]. In the i.i.d setting, the analytical optimal solution to the mean-variance formulation in multiperiod portfolio selection has been also considered by many authors (see e.g. Li et al. [2], Li and Ng[3] and Samuelson[4]. In this section, we consider a capital market with with $(n + 1)$ risky securities, with random rate of returns. An investor joins the market at time 0 an initial wealth x_0. The investor can allocate among the $(n + 1)$ assets. The rate of risky securities at time period t are denoted by a vector $[e_t = e_t^0, e_t^1, \ldots, e_t^n]'$, where e_t^i is random return for securities at time period t are denoted by a vector t. Return $\boldsymbol{e}_t$ has a known mean $E(\boldsymbol{e}_t)$ has a known mean $E(e_t) = Ee_t^0, Ee_t^1, \ldots, Ee_t^n$ and a known convariance

$$\mathrm{cov}(\mathrm{e_t}) = \begin{pmatrix} \sigma_{t,00} & \cdots & \sigma_{t,0n} \\ \vdots & \ddots & \vdots \\ \sigma_{t,0n} & \cdots & \sigma_{t,nn} \end{pmatrix}$$

Let x_t be the wealth of investor at the beginning of the t the period., and let $u_t^i, i = 1, 2, \ldots, n,$ be amount invested in the i th riskey asset at beginning of the t th time period. The amount invested in the 0 th riskey asset at the beginning of the t th time period is equal to $x_t = -\sum_{i=1}^n u_t^i$. An investor is seeking a best investment strategy, $\boldsymbol{u}_t = \left[\boldsymbol{u}_t^1, \boldsymbol{u}_t^2, \ldots \boldsymbol{u}_t^n\right]'$ for $t = 0,1,2,\ldots T\text{-}1$, such that (i) the expected value of the terminal wealth x_t, $E(x_t)$, is maximized if the variance of terminal wealth, $\mathrm{Var}(x_t)$, is not greater than prescribed risk level, or (ii) the variance of terminal wealth, $\mathrm{Var}(x_t)$, is maximized if expected terminal wealth, $E(x_T)$, is not smaller than a prescibed level. Mathmatically, a mean-variance formulation for multiperiod portfolio selection can be posed as one of following two forms:

$$\begin{aligned}(P1(\sigma)) : &\max \mathrm{E}(x_{T_{,,}}) \\ &\text{s.t } \mathrm{Var}(x_T) \le \sigma \\ &x_{t+1} = \sum_{i=1}^n e_t^i u_t^i + (x_t - \sum_{i=1}^n u_t^i) e_t^0 \\ &= e_t^0 x_t + p_t' u_t \quad t = 0, 1, \ldots, T-1 \end{aligned} \qquad (1)$$

$$\begin{aligned}(P2(\in)) : &\min \mathrm{Var}(x_T) \\ &\text{s.t } \mathrm{E}(x_T) \ge \in \\ &x_{t+1} = \sum_{i=1}^n e_t^i u_t^i + \left(x_t - \sum_{i=1}^n u_t^i\right) e_t^0 \\ &= e_t^0 x_t + P_t' u_t \quad t = 0, 1, \ldots, T-1 \end{aligned} \qquad (2)$$

Where

$$P_t = [p_t^1, p_t^2, \ldots, p_t^n]' = [(e_t^1 - e_t^0), (e_t^2 - e_t^0), \ldots, (e_t^n - e_t^o)]'. \quad (3)$$

Notice that $\mathrm{E}(e_t e_t') = \mathrm{Cov}(e_t) + \mathrm{E}(e_t) E(e_t')$. We assume that $\mathrm{E}(e_t e_t')$ is positive definite for all time periods, that is,

$$\mathrm{E}(e_t e_t') \begin{bmatrix} \mathrm{E}\{(e_t^0)^2\} & \mathrm{E}(e_t^0 e_t^1) & \ldots & \mathrm{E}(e_t^0 e_t^n) \\ \mathrm{E}(e_t^0 e_t^1) & \mathrm{E}\{(e_t^1)^2\} & \ldots & \mathrm{E}(e_t^0 e_t^n) \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{E}(e_t^0 e_t^n) & \mathrm{E}(e_t^n e_t^1) & & \mathrm{E}\{(e_t^n)^2\} \end{bmatrix} > 0 \quad (4)$$

The following holds from equation (4):

$$\begin{bmatrix} \mathrm{E}\{(e_t^0)^2\} & \mathrm{E}(e_t^0 P_t') \\ \mathrm{E}(e_t^0 P_t) & \mathrm{E}(P_t P_t') \end{bmatrix} = \begin{bmatrix} 1 & o & \ldots & 0 \\ -1 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ -1 & 0 & \ldots & 1 \end{bmatrix} \mathrm{E}(e_t e_t') \begin{bmatrix} 1 & -1 & \ldots & -1 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

$$\forall t = 0, 1, \ldots. T\text{-}1 \qquad (5)$$

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Forthermore, we have the followings from equation (5):

$$\mathrm{E}(P t P_t^{'})>0, \qquad \forall t=0,1,....T\text{-}1 \tag{6}$$

and

$$\mathrm{E}((e_t^0)^2)-\mathrm{E}(e_t^0P_t^{'})\mathrm{E}^{-1}(P_tP_t^{'})\mathrm{E}(e_t^0P_t)>0 \quad \forall t=0,1,....T\text{-}1 \tag{7}$$

An equivalent formulation to either ($P1(\sigma)$) or($P2(\epsilon)$) in generating efficient multiperiod portfolio policies is

$$(\mathrm{E}(\omega)): \max\ \mathrm{E}(x_t)\text{-}\omega\mathrm{var}(x_t)$$

$$\text{s.t}\ x_T=e_t^0x_t+P_t^{'}u_t \quad t=0,1,2,...,T-1 \tag{8}$$

All three problem ($P1(\sigma)$),($P2$ (ϵ)), and (E(ω)) are difficuties to solve directly. The optimal multireriod portfolio policy for problem (E(ω)) will firest be derived. The solution to problem ($P1(\sigma)$) and($P2$ (ϵ)) will then be obtained based on relationships between ($P1(\sigma)$),($P2$ (ϵ)), and (E(ω)). Define $\Pi_{\mathrm{E}}(\omega)$ to be the set of optimal solutions of problem (E(ω)) with given ω, that is

$$\Pi_{\mathrm{E}}(\omega)=\{\pi\,|\,\pi \text{ is maximizer of } (\mathrm{A}(\mathrm{E},\omega))\}. \tag{9}$$

Define

$$\begin{aligned}&U(\mathrm{E}(x_T^2)),\mathrm{E}(x_T))\\&=\mathrm{E}(x_T)\text{-}\omega\mathrm{Var}(x_T)\\&=-\omega\mathrm{E}(x_T^2)+[\omega\mathrm{E}^2(x_T)+\mathrm{E}(x_T)].\end{aligned} \tag{10}$$

It is obvious that $\widetilde{U}$ is a convex function of $\mathrm{E}(x_T^2)$ and $\mathrm{E}(x_T)$. The following auxiliary problem is now constructed for E(ω),

$$\begin{aligned}&(\mathrm{A}(\lambda,\omega): \max\ \mathrm{E}\{\text{-}\omega x_T^2+\lambda x_T\}\\&\qquad \text{s.t}\ x_{t+1}=e_t^0x_t+P_t^{'}u_t \quad t=0,1,2,...,T-1.\end{aligned} \tag{11}$$

Define $\Pi_A(\lambda, \omega)$ to be the set of optimal solutions of problem (A (λ, ω)) with given λ and ω, that is

$$\Pi_A(\lambda,\mathrm{w})=\{\pi\,|\,\pi \text{ is maximizer of } (\mathrm{A}(\lambda,\omega))\}. \tag{12}$$

Denote

$$\begin{aligned}d(\pi,\omega)&=\frac{\partial U(E(x_T^2),E(x_T))}{\partial\mathrm{E}(x_T)}\Big|_\pi\\&=1+2\omega\mathrm{E}(x_T)|_\pi\end{aligned} \tag{13}$$

Now we can introduce the fillowing results (see also Reid[5])

Lemma 1: *For any* $\pi^* \in \Pi_E(\omega)$, $\pi^* \in \Pi_A(d(\pi^*, \omega)\omega)$.

Lemma 2: *Assume* $\pi^* \in \Pi_A(\lambda^*,\omega)$. A neceassry condition for $\pi^* \in \Pi_E(\omega)$ is $\lambda^* = 1 + 2\,\omega\mathrm{E}(x_T)|_\pi$

The optimal solution of auxiliary problem ($A(\lambda, \omega)$) can be derived analytically using dynamic programing. The dynamic programming algorithim starts from stage T-1. For given $x_{T-1,}$ the optimization problem is given as follow

$$\begin{aligned}&\max J(u_{T-1}\,|\,x_{T-1})\\&=\max\ \mathrm{E}\{\text{-}\omega x_T^2+\lambda x_T\}\\&=\max\{\omega\mathrm{E}\{e_{T-1}^0\}^2\}x_{T-1}^2+\lambda\mathrm{E}(e_{T-1}^0)x_{T-1}\}\\&+\{\lambda\mathrm{E}(P_{T-1}^{'})-2\omega x_{T-1}\mathrm{E}(e_T^0+P_{T-1}^{'})u_{T-1}-\omega u_{T-1}^{'}E(P_{T-1}P_{T-1}^{'})u_{T-1}.\end{aligned} \tag{14}$$

Optimal $u_{T\text{-}1}$ can be obtained by solving $\dfrac{dJ(u_{T-1}\,|\,x_{T-1})}{du_{T-1}}=0$ with

$$u_{T-1}^*=\mathrm{E}^{-1}(P_{T-1}P_{T-1}^{'})[\mathrm{E}(P_{T-1})\frac{\lambda}{2\omega}-\mathrm{E}(e_{T-1}^0P_{T-1})x_{T-1}]. \tag{15}$$

Substituting u_{T-1}^* back to $J_{T-1(x_{t-1})}$, we have the optimal cost-to-go at given $x_{T\text{-}1.}$

$$\begin{aligned}J_{T-1}^*(x_{T-1})&=-\omega[\mathrm{E}\left(\left(e_{T-1}^0\right)^2-\mathrm{E}(e_{T-1}^0P_{T-1}^{'})\mathrm{E}^{-1}(P_{T-1}P_{T-1}^{'})\mathrm{E}(e_{T-1}^0P_{T-1})x_{T-1}^2\right.\\&+\lambda[\mathrm{E}(e_{T-1}^0)-\mathrm{E}(P_{T-1}^{'})\mathrm{E}^{-1}(P_{T-1}P_{T-1}^{'})\mathrm{E}(e_{T-1}^0P_{T-1})]x_{T-1}\\&+\frac{\lambda^2}{4\omega}\mathrm{E}(P_{T-1}^{'})\mathrm{E}^{-1}(P_{T-1}P_{T-1}^{'})E(P_{T-1})\end{aligned} \tag{16}$$

The derived utility function has a similar form at stage t, $0 \le t \le T$-1, to the original utility function has a similar form at staget T. We can derive the optimal portfolio decision and the optimal cost-to-go for given x_t at stage t, $0 \le t \le T$-2, in a similar manner,

$$\begin{aligned}u_t^*(x_t)&=\mathrm{E}^{-1}(P_tP_t^{'})\mathrm{E}(P_t)\frac{\lambda\prod_{K=t+1}^{T-1}(E(e_k^0)\mathrm{E}^{-1})(P_kP_k^{'})\mathrm{E}(e_k^0P_k)}{2\omega\prod_{k=t+1}^{T-1}(\mathrm{E}(e_k^0)^2-\mathrm{E}(e_k^0P_k^{'})\mathrm{E}^{-1}(P_kP_k^{'})E(e_k^0P_k)}\\&\text{-}\mathrm{E}^{-1}(P_tP_t^{'})\mathrm{E}(e_t^0P_t)x_t\end{aligned} \tag{17}$$

and

$$\begin{aligned}J_T^*(x_T)&=-\omega[\mathrm{E}(\left(e_T^0\right)^2-\mathrm{E}(e_T^0P_T^{'})\mathrm{E}^{-1}(P_TP_T^{'})\mathrm{E}(e_T^0P_T)x_T^2\\&+\lambda[\mathrm{E}(e_T^0)-\mathrm{E}(P_T^{'})\mathrm{E}^{-1}(P_TP_T^{'})\mathrm{E}(e_T^0P_T)]x_T\\&+\frac{\lambda^2}{4\omega}\mathrm{E}(P_T^{'})\mathrm{E}^{-1}(P_TP_T^{'})E(P_T).\end{aligned} \tag{18}$$

Analytical Solution for time series

Time series return process in econometric modeling have been considered mainly in signale period portfolio selection problem (see e.g. Gourieroux [6] and Gourieroux and Jasiak [7]. In this section, we consider the optimal portfolio policy for auxiliary problem ($A(\lambda,w)$) at each time period t is of the following form

$$u_t^*(x_t;\gamma)=\text{-}K_tx_t+v_t(\gamma) \qquad t=0,1,...T\text{-}1 \tag{19}$$

Where

$$\tilde{a}=\frac{\lambda}{\omega} \tag{20}$$

$$K_t=\mathrm{E}^{-1}(P_tP_t^{'})\mathrm{E}(e_t^0P_t) \tag{21}$$

$$\upsilon_t(\gamma)=\frac{\gamma}{2}\left(\prod_{k=t+1}^{T\text{-}1}\frac{A_k^1}{A_k^2}\right)\mathrm{E}^{-1}(P_tP_t^{'})\mathrm{E}(P_t) \qquad t=0,1,2,...,T\text{-}1 \tag{22}$$

$$A_k^1=\mathrm{E}(e_k^0)\text{-}\mathrm{E}(P_k^{'})\mathrm{E}^{-1}(P_kP_k^{'})\mathrm{E}(e_k^0P_k) \tag{23}$$

$$A_k^2=\mathrm{E}(e_k^0)^2)-\mathrm{E}(e_k^0P_k^{'})\mathrm{E}^{-1}(P_kP_k^{'})\mathrm{E}(e_k^0P_k) \tag{24}$$

with the following boundary condition

$$\upsilon_{T\text{-}1}(\gamma)=\frac{\gamma}{2}\ \mathrm{E}^{-1}(P_{T\text{-}1}P_{T\text{-}1}^{'})\ \mathrm{E}(P_{T\text{-}1}) \tag{25}$$

Wealth of investor is expressed as recursive from substituting u_t^* into $x_{T\text{-}1}$

$$x_{T+1}(\gamma)=(e_t^0\text{-}P_t^{'}K_t)x_t(\gamma)+P_t^{'}v_t(\gamma). \tag{26}$$

Squared on both sides of (26) yields

$$\begin{aligned}x_{t+1}^2(\gamma)&=[(e_t^0)^2]\text{-}2e_t^0P_t^{'}K_t+K_t^{'}P_tP_t^{'}K_t]x_t^2(\gamma)\\&+2(e_t^0\text{-}P_t^{'})x_t(\gamma)P_t^{'}\upsilon_t(\gamma)+\upsilon_t(\gamma)^{'}P_tP_t^{'}\upsilon_t(\gamma).\end{aligned} \tag{27}$$

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Than, we take expected values and substitute time series return process to get time series effect on the optimal solution of multiperiod portfolio.

First, we consider MA(1) model

$$P_t = \tilde{P}_t + B\tilde{P}_{t-1} + \mu_t. \tag{28}$$

We assume $E(\tilde{P}_t)=0$, $\tilde{P}_t$ are mutually independent, so that

$$E(\tilde{P}_{t-i}\tilde{P}_{t-j}) = E(\tilde{P}_{t-i})\,E(\tilde{P}_{t-j}),\ i \neq j, \tag{29}$$

x_t is F_t-measurable and P_t is independent of F_t (P_t is not dependent). We take expections on both side of wealth of investor (26)

$$\begin{aligned} E(x_{t+1}(\gamma)) &= E((e_t^0 - P_t^{'}K_t)x_t(\gamma) + P_t^{'}v_t(\gamma)) \\ &= E(e_t^0 x_t) - E(x_t P_t^{'})K_t + E(P_t^{'})v_t \end{aligned} \tag{30}$$

Taking expection on both sides of (27), we have

$$\begin{aligned} E(x_{t+1}^2(\gamma)) &= E((e_t^0)^2 - 2e_t^0 P_t^{'}K_t + K_t^{'}P_tP_t^{'}K_t) \\ &\quad + 2(e_t^0 - P_t^{'}K_t)x_t(\gamma)P_t^{'}v_t(\gamma) + v_t(\gamma)^{'}P_tP_t^{'}v_t(\gamma)) \\ &= E((e_t^0)^2 x_t^2) - 2E(e_t^0 x_t^2 P_t^{'})K_t + K_t^{'}E(P_tP_t^{'})K_t \\ &\quad + 2(E(e_t^0 x_t P_t^{'}) - K_t E(P_tP_t^{'})K_t)v_t + v_t^{'}E(P_tP_t^{'})v_t. \end{aligned} \tag{31}$$

Substituting time series $P_t = \tilde{P}_t + B\tilde{P}_{t-1} + \mu_t$ $E(x_{t+1}(\gamma))$ and $E(x_{t+1}^2(\gamma))$ we have

$$E(x_{t+1}(\gamma)) = E(e_t^0 x_t) - E(P_t^{'})E(x_t) - E(x_t\tilde{P}_{t-1}^{'})B^{'} - E(x_t\mu_t^{'}) + E(P_t^{'})v_t \tag{32}$$

$$\begin{aligned} E(x_{t+1}^2(\gamma)) &= E((e_t^0)^2 x_t^2) - 2(E(e_t^0 x_t^2)E(\tilde{P}_t^{'}) + E(e_t^0\tilde{P}_{t-1}^{'}x_t^2)B^{'} + E(e_t^0\mu_t^{'}x_t^2)) \\ &\quad + K_t(E\tilde{P}_t\tilde{P}_t^{'})E(x_t^2) + E(\tilde{P}_t)E(\tilde{P}_{t-1}^{'}x_t^2) + E(\tilde{P}_t)E(\mu_t^{'}x_t^2)) \\ &\quad + BE(\tilde{P}_{t-1}^{'}x_t^2)E(\tilde{P}_t^{'})() + BE(\tilde{P}_{t-1}\tilde{P}_{t-1}^{'}x_t^2)B^{'} + BE(\tilde{P}_{t-1}^{'}x_t^2\mu_t) \\ &\quad + E(\mu_t x_t^2)E(\tilde{P}_t^{'}) + E(\mu_t\tilde{P}_{t-1}x_t^2)B^{'} + E(\mu_t\mu_t^{'}x_t^2))K_t \\ &\quad + 2(E(e_t^0\tilde{P}_t^{'})E(x_t) - E(e_t^0 x_t\tilde{P}_{t-1}^{'})B^{'} - E(e_t^0 x_t\mu_t^{'})) \\ &\quad - K_t(E(\tilde{P}_t\tilde{P}_t^{'})E(x_t) + E(\tilde{P}_t)E(\tilde{P}_t^{'}x_t) + E(\tilde{P}_t)E(\mu_t^{'}x_t) \\ &\quad + BE(\tilde{P}_t^{'}x_t)E(\tilde{P}_t^{'}) + BE(\tilde{P}_{t-1}\tilde{P}_{t-1}^{'}x_t)B^{'} + BE(\tilde{P}_{t-1}^{'}x_t\mu_t) \\ &\quad + E(\mu_t x_t)E(\tilde{P}_t^{'}) + E(\mu_t\tilde{P}_{t-1}x_t)B^{'} + E(\mu_t\mu_t^{'}x_t)))v_t + v_t^{'}E(P\tilde{P}_t^{'})v_t. \end{aligned} \tag{33}$$

Here, $E(\tilde{P}_{t-1}^{'}x_t)$ and $E(\tilde{P}_{t-1}\tilde{P}_{t-1}^{'}x_t)$ are unknown. Let $y_t = \tilde{P}_{t-1}^{'}x_t$, $z_t = \tilde{P}_{t-1}\tilde{P}_{t-1}^{'}x_t$. Then, we constitute recurrence relation of matrix from x_{t-1}, y_{t+1} *and* Z_{t+1}.

Denote

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} H_{t-1} & F_{t-1} & 0 \\ G_{t-1} & J_{t-1} & 0 \\ N_{t-1} & M_{t-1} & 0 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} \delta_{t-1} \\ s_{t-1} \\ \sigma_{t-1} \end{pmatrix}, \tag{34}$$

Where

$$\begin{aligned} & y_t = \tilde{P}_{t-1}^{'}x_t, \quad z_t = \tilde{P}_{t-1}\tilde{P}_{t-1}^{'}x_t \\ & H_t = e_t^0 - (\tilde{P}_t^{'} + \mu_t)K_t,\ F_t = B^{'}K_t,\ \delta_t = P_t^{'}v_t \\ & G_t = (e_t^0 - (\tilde{P}_t^{'} + \mu_t)K_t)\tilde{P}_t^{'},\ J_t = B^{'}K_t\tilde{P}_t^{'},\ s_t = P_t^{'}v_t\tilde{P}_t^{'} \\ & M_t = (e_t^0 - (\tilde{P}_t^{'} + \mu_t)K_t)\tilde{P}_t\tilde{P}_t^{'}),\ N_t = B^{'}K_t\tilde{P}_t\tilde{P}_t^{'},\ \sigma_t = P_t^{'}v_t\tilde{P}_t\tilde{P}_t^{'}. \end{aligned} \tag{35}$$

This final form is given by

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \left(Id - \begin{pmatrix} H_{t-1} & F_{t-1} & 0 \\ G_{t-1} & J_{t-1} & 0 \\ N_{t-1} & M_{t-1} & 0 \end{pmatrix} L \right)^{-1} \begin{pmatrix} \delta_{t-1} \\ s_{t-1} \\ \sigma_{t-1} \end{pmatrix}, \tag{36}$$

where *Id* is unit matrix and *L* is lag operator. Therefore, we see that

Taking expection and simplifying, we have

$$\begin{aligned} E\begin{pmatrix} \delta_{t-1} \\ s_{t-1} \\ \sigma_{t-1} \end{pmatrix} &= E\begin{pmatrix} \tilde{P}_{t-1}^{'}v_{t-1} + \tilde{P}_{t-2}^{'}B^{'}v_{t-1} + \mu_t^{'}v_{t-1} \\ \tilde{P}_{t-1}^{'}v_{t-1}\tilde{P}_{t-1}^{'} + \tilde{P}_{t-2}^{'}B^{'}v_{t-1}\tilde{P}_{t-1}^{'} + \mu_t^{'}v_{t-1}\tilde{P}_{t-1}^{'} \\ \tilde{P}_{t-1}^{'}v_{t-1}\tilde{P}_{t-1}\tilde{P}_{t-1}^{'} + \tilde{P}_{t-2}^{'}B^{'}v_{t-1}\tilde{P}_{t-1}\tilde{P}_{t-1}^{'} + \mu_{t-1}^{'}v_{t-1}\tilde{P}_{t-1}\tilde{P}_{t-1}^{'} \end{pmatrix} \\ &= E\begin{pmatrix} \mu_{t-1}^{'}v_{t-1} \\ \tilde{P}_{t-1}^{'}v_{t-1}\tilde{P}_{t-1}^{'} \\ \tilde{P}_{t-1}^{'}v_{t-1}\tilde{P}_{t-1}\tilde{P}_{t-1}^{'} + \mu_{t-1}^{'}v_{t-1}\tilde{P}_{t-1}\tilde{P}_{t-1}^{'} \end{pmatrix} \end{aligned} \tag{37}$$

Next, we consider MA(2) model

$$P_t = \tilde{P}_t + B_1\tilde{P}_{t-1} + B_2\tilde{P}_{t-2}$$

Let $y_t^1, y_t^2, z_t^1, z_t^2$ and z_t^3 be the following form

$$\begin{aligned} & y_t^1 = \tilde{P}_{t-1}^{'}x_t,\ y_t^2 = \tilde{P}_{t-2}^{'}x_t \\ & z_t^1 = \tilde{P}_{t-1}\tilde{P}_{t-1}^{'}x_t, \quad z_t^2 = \tilde{P}_{t-1}\tilde{P}_{t-2}^{'}x_t,\ z_t^3 = \tilde{P}_{t-2}\tilde{P}_{t-2}^{'}x_t \end{aligned} \tag{38}$$

These are neceassry to solve optimal solution with MA(2) model . Then, similarly in MA(1) case, we can express the following matrix form by x_t, y_t^1, Z_t^2, and Z_t^3

$$\begin{aligned} \begin{pmatrix} x_t \\ y_t^1 \\ y_t^2 \\ z_t^1 \\ z_t^2 \\ z_t^3 \end{pmatrix} &= \begin{pmatrix} H_{t-1} & F_{t-1}^1 & F_{t-1}^2 & 0 & 0 & 0 \\ G_{t-1} & J_{t-1}^{y1} & J_{t-1}^{y2} & 0 & 0 & 0 \\ 0 & J_{t-1}^{y3} & 0 & J_{t-1}^{z1} & J_{t-1}^{z2} & 0 \\ N_{t-1}^1 & M_{t-1}^{y1} & M_{t-1}^{y2} & 0 & 0 & 0 \\ N_{t-1}^2 & M_{t-1}^{y3} & 0 & 0 & 0 & 0 \\ N_{t-1}^3 & 0 & M_{t-1}^{y2} & M_{t-1}^{z7} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1}^1 \\ y_{t-1}^2 \\ z_{t-1}^1 \\ z_{t-1}^2 \\ z_{t-1}^3 \end{pmatrix} + \\ & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ N_{t-2}^2 & M_{t-2}^{y3} & M_{t-2}^{y4} & M_{t-2}^{z4} & M_{t-2}^{z5} & 0 \\ N_{t-2}^3 & M_{t-2}^{y5} & M_{t-2}^{y6} & M_{t-2}^{z7} & M_{t-2}^{z8} & 0 \end{pmatrix} \begin{pmatrix} x_{t-2} \\ y_{t-2}^1 \\ y_{t-2}^2 \\ z_{t-2}^1 \\ z_{t-2}^2 \\ z_{t-2}^3 \end{pmatrix} + \begin{pmatrix} \delta_t \\ s_t^1 \\ s_t^2 \\ \sigma_t^1 \\ \sigma_t^2 \\ \sigma_t^3 \end{pmatrix}. \end{aligned} \tag{39}$$

Further Discussion

In this section, we consider the general MA(t) model

$$P_t = \tilde{P}_t + B_{(1)}\tilde{P}_{t-1} + \ldots + B_{(t)}\tilde{P}_0 = \sum_{i=0}^{t} B_{(i)}\tilde{P}_{t-i} \tag{40}$$

Substituting $P_t = \sum_{i=0}^{t} B_{(i)}\tilde{P}_{t-i}$ into x_{t+1}, we have

$$\begin{aligned} x_{t+1} &= (e_t^0 - K_t^{'}P_t)x_t + P_tv_t \\ &= (e_t^0 - K_t^{'}\sum_{i=0}^{t} B_{(i)}\tilde{P}_{t-1})x_t + P_tv_t \end{aligned} \tag{41}$$

We need $E(\tilde{P}_{t-i}^{'}x_{t-k})$ and $E(\tilde{P}_{t-i}\tilde{P}_{t-j}^{'}x_{t-k})$ to solve the optimal solution of multiperiod portfolio. Let y_{t-k}^{i} and $Z_{t-k}^{i,j}$ be given by the follwing equations

$$y_{t-k}^{i} = \tilde{P}_{t-i}^{'}x_{t-k} \tag{42}$$

$$y_{t-k}^{(i,j)} = \tilde{P}_{t-i}\tilde{P}_{t-j}^{'}x_{t-k}. \tag{43}$$

Then, we see that

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$$x_t \begin{pmatrix} \tilde{P}_{t-1}' \\ \tilde{P}_{t-2}' \\ \vdots \\ \tilde{P}_0' \end{pmatrix} = (e^0_{t-1} - K_{t-1}' \sum_{i=0}^{t-1} B_{(i)} \tilde{P}_{t-i-1}) \begin{pmatrix} \tilde{P}_{t-1}' \\ \tilde{P}_{t-2}' \\ \vdots \\ \tilde{P}_0' \end{pmatrix} + P_{t-1}' v_{t-1} \begin{pmatrix} \tilde{P}_{t-1}' \\ \tilde{P}_{t-2}' \\ \vdots \\ \tilde{P}_0' \end{pmatrix} =$$

$$e_{t-1} \begin{pmatrix} \tilde{P}_{t-1}' \\ \tilde{P}_{t-2}' \\ \vdots \\ \tilde{P}_0' \end{pmatrix} - K_{t-1} \begin{pmatrix} \tilde{P}_{t-1}\tilde{P}_{t-1}' x_{t-1} + \sum_{i=2}^{t-1} B_{(i)} y^i_{t-1} \tilde{P}_{t-1}' \\ \sum_{i=1}^{t-1} B_{(i)} z^{(i,2)}_{t-1} \\ \vdots \\ \sum_{i=1}^{t-1} B_{(i)} z^{(i,t-1)}_{t-1} \end{pmatrix} + \begin{pmatrix} \sigma^1 \\ \sigma^2 \\ \vdots \\ \sigma^t \end{pmatrix} \quad (44)$$

$$\sigma^s = (v_{t-1}' \sum_{i=0}^{t-1} B_{(i)} \tilde{P} \tilde{P}'_{t-s}). \quad (45)$$

Taking expectation and simplyfiing, we have

$$E(\sigma^s) = E(v_{t-1}' \sum_{i=0}^{t-1} B_{(i)} \tilde{P}_{t-i-1} \tilde{P}'_{t-s}) = \upsilon_{t-1}' E(B_{(s-1)} \tilde{P}_{t-s} \tilde{P}'_{t-s}). \quad (46)$$

In this case, we see that

$$x_t \begin{pmatrix} \tilde{P}_{t-1} \\ \tilde{P}_{t-2} \\ \vdots \\ \tilde{P}_0 \end{pmatrix} \left(\tilde{P}_{t-1}' \; \tilde{P}_{t-2}' \ldots \tilde{P}_0' \right)$$

$$= (e^0_{t-1} - \tilde{P}_{t-1}' K_{t-1}) \begin{pmatrix} \tilde{P}_{t-1}\tilde{P}_{t-1}' x_{t-1} & \tilde{P}_{t-1}' y^2_{t-1} & \cdots & \tilde{P}_{t-1}' y^t_{t-1} \\ y^2_{t-1} \tilde{P}_{t-1}' & z^{(2,2)}_{t-1} & \cdots & z^{(2,2)}_{t-1} \\ \vdots & \vdots & \ddots & \vdots \\ y^t_{t-1} \tilde{P}_{t-1}' & z^{(t,2)}_{t-1} & \cdots & z^{(t,t)}_{t-1} \end{pmatrix} \quad (47)$$

$$-K_{t-1}' \begin{pmatrix} \sum_{k=1}^{t} B_{(k)} \tilde{P}_{t-k-1} \tilde{P}_{t-1} \tilde{P}_{t-1}' x_{t-1} & \cdots & \sum_{k=1}^{t} B_{(k)} \tilde{P}_{t-k-1} \tilde{P}_{t-1} \tilde{P}_0' x_{t-1} \\ \vdots & & \vdots \\ \sum_{k=1}^{t} B_{(k)} \tilde{P}_0 \tilde{P}_{t-1} \tilde{P}_{t-1}' x_{t-1} & \cdots & \sum_{k=1}^{t} B_{(k)} \tilde{P}_{t-k-1} \tilde{P}_0 \tilde{P}_0' x_{t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} \delta_{(1,1)} & \cdots & \delta_{(1,t)} \\ \vdots & \ddots & \vdots \\ \delta_{(t,1)} & \cdots & \delta_{(t,t)} \end{pmatrix}.$$

Now we can not transform this matrix into easy to solve expression like substituting MA(1) model or MA (2) model. So we need to transform this matrix into expression with independent coefficients, x_{t-1}, y^i_{t-k} and $Z^{i,j}_{t-k}$.

This problem will be left for the further consideration.

We also have to consider the numerical study to demonstrate the adoption of the multiperiod mean-variance formulations for time series return processes and the efficiancy of the solution methods derived in this paper.

We would like to derive optimal solution case that we have an additional restriction, namely casuality from another index to apply our method to pension investment problem, which will be also the further work.

Competing Interests

The authors declare that they have no competing interests.

Author Contributions

All the authors substantially contributed to the study conception and design as well as the acquisition and interpretation of the data and drafting the manuscript.

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